Gravitational Perturbation of Traversable Wormholes and Stability

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Previous Studies on the existance of stable wormholes

- M-AdS wormholes with negative mass : unstable under radial perturbation
- Black Universe : stable with certain parameter

K.A. Bronnikov, R.A. Konoplya and A. Zhidenko, Instabilities of wormholes and regular blackholes supported by a phantom scalar eld, Phys. Rev. D 86 (2012) 024028 [arXiv:1205.2224]

- Thin-shell wormholes : the Gauss-Bonnet term destabilizes the stability of the spherically symmetric wormholes

T. Kokubu, H. Maeda and T. Harada, Does the Gauss-Bonnet term stabilize wormholes, Class. Quant. Grav. 32 (2015) 235021 [arXiv:1506.08550]

- Thin-Shell wormholes with a phantom-like equation of state
  - Schwarzschild wormholes : unstable
  - Wormholes with a cosmological constant
    - Schwarzschild-de Sitter spacetime : stable
    - Schwarzschild-anti de Sitter spacetime : stable
  - Reissner-Nordstrom wormholes : unstable
  - Thin-shell wormholes from regular charged BHs : unstable
  - Thin-shell wormholes based on dilaton and dilaton-axion black holes : unstable

P.K.F. Kuhttig, The Stability of thin-shell wormholes with a phantom-like equation of state, Acta Phys. Polon. B 41 (2010) 2017 [arXiv:1008.3111]

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Gravitational Waves

- General Relativity
  - Gravity = Curvature of spacetime
  - Mass determines the curvature of spacetime.
  - The curvature tells mass how to move.

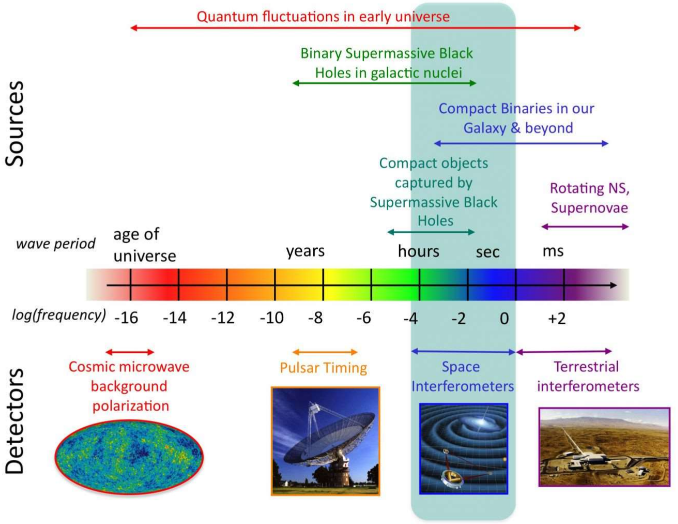
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(|h_{\mu\nu}|^2)$$

$\eta_{\mu\nu}$: flat Minkowski spacetime metric, (-1, 1, 1, 1)
$h_{\mu\nu}$: perturbed term ($h_{\mu\nu} \ll 1$)

S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

The Gravitational Wave Spectrum



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Morris-Thorne Wormhole

M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).

$$ds^2 = -e^{2\Phi(r)}dt^2 + dl^2 + r(l)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -e^{2\Phi(r)}dt^2 + \frac{dr^2}{(1-\frac{b(r)}{r})} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$\Phi(r)$: red-shift function

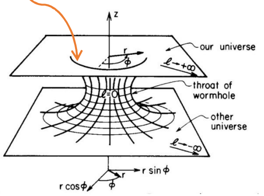
$b(r)$: shape function

l : Proper radial distance

$$\frac{dr}{dl} = \pm\sqrt{1-\frac{b(r)}{r}}$$

$$\frac{d^2r}{dl^2} > 0 \Rightarrow b(r) - rb'(r) > 0$$

Flaring-out condition



Properties of Traversable Wormholes

- Spherically symmetric and static
- Obey Einstein equations everywhere
- Must have a throat
- No horizons
- Stable in perturbations

Morris-Thorne Wormhole

M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).

- $ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{(1-\frac{b(r)}{r})} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta})$
- $R^{\delta}_{\mu\lambda\nu} = \Gamma^{\delta}_{\mu\nu,\lambda} - \Gamma^{\delta}_{\mu\lambda,\nu} + \Gamma^{\delta}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\delta}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}$
- **Non-Vanishing components**

$\Gamma^t_{tr} = \Phi'$	$\Gamma^r_{\phi\phi} = -(r-b)\sin^2\theta$
$\Gamma^r_{tt} = e^{2\Phi}\Phi'\left(1-\frac{b}{r}\right)$	$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = r^{-1}$
$\Gamma^r_{rr} = \frac{br'-b}{2r(r-b)}$	$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$
$\Gamma^r_{\theta\theta} = -(r-b)$	$\Gamma^\phi_{\theta\phi} = \cot\theta$

$R_{tt} = e^{2\Phi}\left[\left(\Phi'' + (\Phi')^2 + \frac{2}{r}\Phi'\right)\left(1-\frac{b}{r}\right) - \frac{b'r-b}{2r^2}\Phi'\right]$
$R_{rr} = -\Phi'' - (\Phi')^2 + \frac{b'r-b}{2r(r-b)}\left(\Phi' + \frac{2}{r}\right)$
$R_{\theta\theta} = \frac{b'r+b}{2r} - (r-b)\Phi'$
$R_{\phi\phi} = \sin^2\theta\left[\frac{b'r+b}{2r} - (r-b)\Phi'\right]$

Morris-Thorne Wormhole

M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).

- **Orthogonal basis**

$e_{\hat{t}} = e^{-\Phi}e_t$	$e_{\hat{\theta}} = \frac{1}{r}e_\theta$
$e_{\hat{r}} = \sqrt{1-\frac{b}{r}}e_r$	$e_{\hat{\phi}} = \frac{1}{r\sin\theta}e_\phi$

- **Einstein Tensor, $G_{\mu\nu}$**

$G_{\hat{t}\hat{t}} = \frac{b'}{r}$
$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}$
$G_{\hat{\theta}\hat{\theta}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r}\right] - \frac{b'r-b}{2r^2}\left(\Phi' + \frac{1}{r}\right)$
$G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}}$

- **Stress-energy Tensor, $T_{\mu\nu}$**

$T_{\hat{t}\hat{t}} = \rho(r)$	The total density
$T_{\hat{r}\hat{r}} = -\tau(r)$	The radial tension
$T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r)$	The pressure

Morris-Thorne Wormholes

- Einstein field equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$\rho = \frac{b'}{8\pi G\, r^2}$$

$$\tau = \frac{1}{8\pi G}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right]$$

$$p = \frac{1}{8\pi G}\left[\left(1-\frac{b}{r}\right)\left\{\Phi'' + (\Phi')^2 + \frac{\Phi'}{r}\right\} - \frac{b'r-b}{2r^2}\left(\Phi' + \frac{1}{r}\right)\right]$$

$$b(r) = b(r_0) + \int_{r_0}^{r} 8\pi G\, \rho(r') r'^2\, dr'$$

$b_{\pm} = 2M_{\pm}$ → **Shape function is closely relate to the distribution of mass inside the wormhole**

At the throat, flaring-out condition ⇒ $\tau - \rho > 0$ "exotic matter" $\tau > \rho > 0$

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Linearized Theory of Gravity

S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

- Weak field approximation

  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(|h_{\mu\nu}|^2)$

  $\eta_{\mu\nu}$: flat Minkowski spacetime
  $h_{\mu\nu}$: perturbed term ($h_{\mu\nu} \ll 1$)

  S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

- Traversable wormhole spacetime is not vacuum.

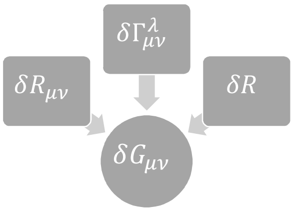
  $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

  Background spacetime is curved spacetime

  $G_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + \delta T_{\mu\nu}\right)$

  $\nabla_\nu T^{\mu\nu} = 0$

- $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}\left(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right)$
- $R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\delta_{\lambda\delta} - \Gamma^\lambda_{\mu\delta}\Gamma^\delta_{\nu\lambda}$
- $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$
- Perturbed terms



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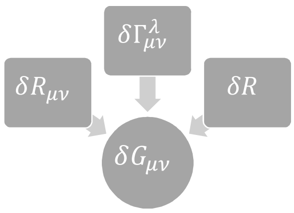
Linearized Theory of Gravity

S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

- Perturbed terms
- $\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta\Gamma^\lambda_{\mu\nu}$
- $\bar{R}_{\mu\nu} = R_{\mu\nu} + \delta R_{\mu\nu}$
- $\bar{R} = R + \delta R$
- $\bar{G}_{\mu\nu} = G_{\mu\nu} + \delta G_{\mu\nu}$
- Linearized Einstein Tensor
- $\bar{G}_{\mu\nu} = G_{\mu\nu} + \delta G_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{\bar{R}}{2}\bar{g}_{\mu\nu}$



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Linearized Theory of Gravity

- Perturbed spacetime

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}h^{\lambda\delta}(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta})$$
$$+\frac{1}{2}g^{\lambda\delta}(h_{\mu\delta,\nu} + h_{\nu\delta,\mu} - h_{\mu\nu,\delta})$$
$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\delta}_{\lambda\delta} + \Gamma^{\delta}_{\lambda\delta}\delta\Gamma^{\lambda}_{\mu\nu}$$
$$-\Gamma^{\lambda}_{\mu\delta}\delta\Gamma^{\delta}_{\nu\lambda} - \Gamma^{\delta}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\delta}$$
$$\delta R = g^{\mu\nu}\delta R_{\mu\nu} - h^{\mu\nu}R_{\mu\nu}$$
$$\delta G_{\mu\nu}$$
$$= \delta R_{\mu\nu} - \frac{R}{2}h_{\mu\nu} - \frac{1}{2}(g^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta})g_{\mu\nu}$$

- Energy Conservation

$$\nabla_\nu T^{\mu\nu} = 0$$

∇_ν : covariant derivative

$$\nabla_\lambda T^{\mu\nu} = \partial_\lambda T^{\mu\nu} + \Gamma^{\mu}_{\lambda\delta}T^{\delta\nu} + \Gamma^{\nu}_{\lambda\delta}T^{\delta\mu}$$
$$\nabla_\nu(T^{\mu\nu} + \delta T^{\mu\nu}) = 0$$
$$\nabla_\nu T^{\mu\nu} \neq 0,$$
$$\nabla_\nu \delta T^{\mu\nu} \neq 0$$
$$\delta\Gamma^{\mu}_{\nu\delta}T^{\delta\nu} + \delta\Gamma^{\nu}_{\nu\delta}T^{\delta\mu}$$
$$+\partial_\nu\delta T^{\mu\nu} + \Gamma^{\mu}_{\nu\delta}\delta T^{\delta\nu} + \Gamma^{\nu}_{\nu\delta}\delta T^{\delta\mu} = 0$$

Non-stationary and Non-axisymmetric Spacetime

- The Perturbation of spherically symmetric spacetime
  - Only the axisymmetric modes of perturbation are needed
  - Non-axisymmetric modes can be obtained through suitable rotation

    $\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)$, Θ : the rotation polar angle
  - Separable in all four variables : $t,\ r,\ \theta,\ \phi$ — Axial equation (θ, ϕ); Radial equation (t, r)
  - The dependence of an axisymmetric mode on θ and ϕ

$$P_l(\cos\theta) = \sum_{m=-l}^{m=+l} P_l^m(\cos\theta)e^{im\phi}P_l^m(\cos\theta')e^{-im\phi}$$

- Perturbed Spacetime: non-stationary and non-axisymmetric

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - q_r dr - q_\theta d\theta - \sigma dt)^2 + e^{2\mu_r}dr^2 + e^{2\mu_\theta}d\theta^2$$

Traversable Wormholes (Background)

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{(1-\frac{b(r)}{r})} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - q_r dr - q_\theta d\theta - \sigma dt)^2 + e^{2\mu_r}dr^2 + e^{2\mu_\theta}d\theta^2$$

- $e^{2\nu} = e^{2\Phi(r)}$, (i.e. $\nu = \Phi(r)$)
- $e^{2\psi} = r^2 \sin^2\theta$
- $e^{2\mu_\theta} = r^2$
- $\sigma = q_r = q_\theta = 0$

- Gravitational perturbation leads $\sigma = q_r = q_\theta \neq 0$ and other values to be small increments, $\delta\nu, \delta\psi, \delta\mu_r, and\ \delta\mu_\theta$.

Gravitational Perturbations

- Axial Perturbation
- Polar Perturbation

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - q_r dr - q_\theta d\theta - \sigma dt)^2 + e^{2\mu_r}dr^2 + e^{2\mu_\theta}d\theta^2$$

Axial Perturbation

- Rotation-related
- q_r, q_θ and σ determine the perturbations

$$h_{\mu\nu} = e^{2\psi}\begin{pmatrix} 0 & 0 & 0 & -\sigma \\ 0 & 0 & 0 & -q_r \\ 0 & 0 & 0 & -q_\theta \\ -\sigma & -q_r & -q_\theta & 0 \end{pmatrix}$$

$$\delta G_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \delta G_{t\phi} \\ 0 & 0 & 0 & \delta G_{r\phi} \\ 0 & 0 & 0 & \delta G_{\theta\phi} \\ \delta G_{\phi t} & \delta G_{\phi r} & \delta G_{\phi\theta} & 0 \end{pmatrix}$$

Polar Perturbation

- Non-rotational
- ν, ψ, μ_r and μ_θ determine the perturbations

$e^{ax} \approx 1 + ax$, when $x \ll 1$

$$h_{\mu\nu} = 2\begin{pmatrix} -e^{2\nu}\delta\nu & 0 & 0 & 0 \\ 0 & e^{2\mu_r}\delta\mu_r & 0 & 0 \\ 0 & 0 & e^{2\mu_\theta}\delta\mu_\theta & 0 \\ 0 & 0 & 0 & e^{2\psi}\delta\psi \end{pmatrix}$$

$$\delta G_{\mu\nu} = \begin{pmatrix} \delta G_{tt} & \delta G_{tr} & \delta G_{t\theta} & 0 \\ \delta G_{rt} & \delta G_{rr} & \delta G_{r\theta} & 0 \\ \delta G_{\theta t} & \delta G_{\theta r} & \delta G_{\theta\theta} & 0 \\ 0 & 0 & 0 & \delta G_{\phi\phi} \end{pmatrix}$$

Assumptions : $\Phi = 0$, and $b(r) = \frac{b_0^2}{r}$ where b_0 is constant.

Axial Perturbation

- Non-vanishing Einstein tensors : $\delta G_{t\phi}$, $\delta G_{r\phi}$, and $\delta G_{\theta\phi}$.

$$-r^2 \frac{\partial}{\partial t}\left(\frac{\partial q_r}{\partial \mathrm{t}} - \frac{\partial \sigma}{\partial r}\right) + 3\cot\theta\left(\frac{\partial q_r}{\partial \theta} - \frac{\partial q_\theta}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial q_r}{\partial \theta} - \frac{\partial \mathrm{q}_\theta}{\partial \mathrm{r}}\right) = 0$$

$$-r^2 \frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial q_\theta}{\partial t} - \frac{\partial \sigma}{\partial \theta}\right) + \frac{2r^2 - b_0^2}{r}\left(\frac{\partial q_\theta}{\partial r} - \frac{\partial \mathrm{q_r}}{\partial \theta}\right) + (r^2 - b_0^2)\frac{\partial}{\partial r}\left(\frac{\partial q_\theta}{\partial r} - \frac{\partial q_r}{\partial \theta}\right) = 0$$

- Assumption 1 : time dependence follows the Fourier decomposition(Harmonic)

$$t \propto e^{i\omega' t}$$

- Define $Q(r,\theta) = r\sqrt{r^2 - b_0^2}\,\sin^3\theta\left(\frac{\partial q_\theta}{\partial r} - \frac{\partial q_r}{\partial \theta}\right)$
- Assumption 2 : $Q(r,\theta) = R(r)\Theta(\theta) = R(r)C_{l+2}^{-\frac{3}{2}}(\theta)$

  where C_n^ν is the Gegenbauer function.

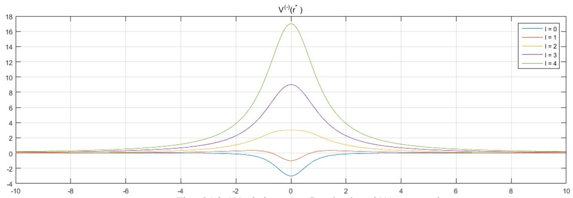
$$C_{l+2}^{-\frac{3}{2}}(\theta) = \left(\frac{\partial^2 \mathrm{P_l}}{\partial \theta^2} - \cot\theta\frac{\partial P_l}{\partial \theta}\right)\sin^2\theta$$

Axial Perturbation

- Radial Equation $r\sqrt{r^2 - b_0^2}\frac{\partial}{\partial r}\left[\frac{\sqrt{r^2-b_0^2}}{r^3}\frac{\partial R}{\partial r}\right] - \frac{(l-1)(l+2)}{r^2}R + \omega'^2 R = 0$
- Coordinate transformation, $\frac{d}{dr^*} = \frac{\sqrt{r^2-b_0^2}}{r}\frac{d}{dr}$
- Assumption 3 : $R(r) = r\,Z^{(-)}(r)$

$$\frac{d^2 Z^{(-)}}{dr^{*2}} + \left(\omega'^2 - V^{(-)}(r)\right)Z^{(-)} = 0$$

$$V^{(-)} = \frac{l(l+1)}{r^2} - \frac{3b_0^2}{r^4}$$



Effective Potential V_{eff}

- The Master Wave Equation
$$\frac{d^2R}{dr_*^2} + \left(\omega^2 - V(r,\omega)\right)R = 0$$
- If the effective potential V_{eff} is positive definite, the differential operater W is a positive self-adjoint operator in the Hilbert space of square integrable functions $L^2(r_*, dr_*)$. Then, there are no negative(growing) mode solutions. All solutions are bounded all of the time.
$$W = -\frac{\partial^2}{\partial r_*^2} + V_{eff}$$
- The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the wormhole, termed the quasinormal modes are damped.

- Quasinormal modes : the eigenvalues of the master wave equation with appropriate boundary conditions : purely incoming waves at the throat and purely outgoing waves at infinity.

Gravitational Perturbations

- Polar Perturbation
$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - q_r dr - q_\theta d\theta - \sigma dt)^2 + e^{2\mu_r}dr^2 + e^{2\mu_\theta}d\theta^2$$
$$ds^2 = -e^{2(\nu+\delta\nu)}dt^2 + e^{2(\mu_r+\delta\mu_r)}dr^2 + e^{2(\mu_\theta+\delta\mu_\theta)}d\theta^2 + e^{2(\psi+\delta\psi)}d\phi^2$$
  - Non-rotational
  - ν, ψ, μ_r and μ_θ determine the perturbations

    $e^{ax} \approx 1 + ax$, when $x \ll 1$

$$h_{\mu\nu} = 2\begin{pmatrix} -e^{2\nu}\delta\nu & 0 & 0 & 0 \\ 0 & e^{2\mu_r}\delta\mu_r & 0 & 0 \\ 0 & 0 & e^{2\mu_\theta}\delta\mu_\theta & 0 \\ 0 & 0 & 0 & e^{2\psi}\delta\psi \end{pmatrix} \qquad \delta G_{\mu\nu} = \begin{pmatrix} \delta G_{tt} & \delta G_{tr} & \delta G_{t\theta} & 0 \\ \delta G_{rt} & \delta G_{rr} & \delta G_{r\theta} & 0 \\ \delta G_{\theta t} & \delta G_{\theta r} & \delta G_{\theta\theta} & 0 \\ 0 & 0 & 0 & \delta G_{\phi\phi} \end{pmatrix}$$

Assumptions : $\Phi = 0$, and $b(r) = \frac{b_0^2}{r}$ where b_0 is constant.

Polar Perturbation

- Non-vanishing Einstein tensors : δG_{tr}, $\delta G_{t\theta}$, $\delta G_{r\theta}$ and $\delta G_{\mu\mu}$.
- Assumption 1: time dependence (Harmonics) $t \propto e^{i\omega' t}$
- Assumption 2: Separation of variables using Legendre function $P_l(\cos\theta)$

$$\nu = N(r)P_l(\cos\theta)$$
$$\mu_r = L(r)\,P_l(\cos\theta)$$
$$\mu_\theta = T(r)\,P_l(\cos\theta) + V(r)P_{l,\theta,\theta}$$
$$\psi = T(r)P_l(\cos\theta) + V(r)\cot\theta\,P_{l,\theta}$$

$$\frac{\partial^2 P_l}{\partial\theta^2} + \cot\theta\frac{\partial P_l}{\partial\theta} = -l(l+1)P_l$$
$$(l-1)(l+2) = 2n$$
$$nV(r) = X(r)$$

- $\delta G_{tr} = \delta G_{t\theta} = \delta G_{r\theta} = 0$

$$\frac{\partial L}{\partial r} + \frac{2}{r}L = -\frac{\partial X}{\partial r} - \frac{1}{r}X$$
$$T - V + L = 0$$
$$\frac{\partial N}{\partial r} - \frac{1}{r}N = \frac{\partial L}{\partial r} + \frac{1}{r}L$$

$N(r), L(r), T(r)$, and $V(r) \Rightarrow N(r), L(r)$ and $V(r)$

- $\delta G_{rr} = 0$

$$\frac{\partial N}{\partial r} - \frac{\partial L}{\partial r} - \frac{\partial X}{\partial r} = \frac{r}{r^2 - b_0^2}\left[(1 - n + r^2\omega'^2)L + (n+1)N + (1 + r^2\omega'^2)X\right]$$

Polar Perturbations

- Three coupled first-order differential equations

⇒ a single second-order differential equation

$$\frac{\partial N}{\partial r} = \alpha N + \beta L + \gamma X$$
$$\frac{\partial L}{\partial r} = \left(\alpha - \frac{1}{r}\right)N + \left(\beta - \frac{1}{r}\right)L + \gamma X$$
$$\frac{\partial X}{\partial r} = -\left(\alpha - \frac{1}{r}\right)N - \left(\beta + \frac{1}{r}\right)L - \left(\gamma + \frac{1}{r}\right)X$$

where

$$\alpha = \frac{(n+1)r}{r^2 - b_0^2}$$
$$\beta = \frac{2b_0^2}{r(r^2 - b_0^2)} - \frac{(n+1)r}{r^2 - b_0^2} + \frac{r^3\omega'^2}{(r^2 - b_0^2)}$$
$$\gamma = \frac{b_0^2}{r(r^2 - b_0^2)} + \frac{r^3\omega'^2}{(r^2 - b_0^2)}$$

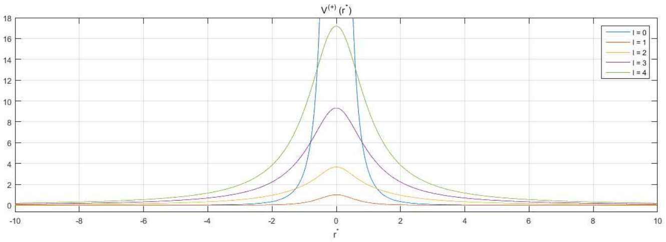
- Define $Z^{(+)}(r) = rV(r) - \frac{r^3}{(nr^2 + b_0^2)}(L(r) + X(r))$

where $r^{*2} = r^2 - b_0^2$ (proper radial distance)

Polar Perturbations

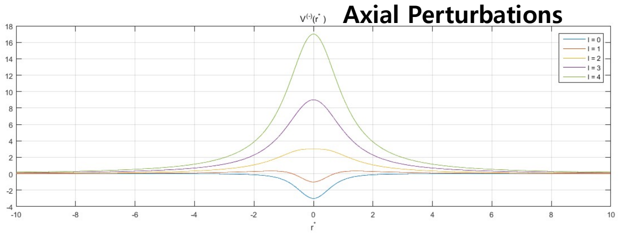
$$\left[\frac{d^2}{dr^{*2}} + \omega'^2\right] Z^{(+)}(r) = V(r) Z^{(+)}(r)$$

where $V^{(+)}(r) = \frac{2n}{r^2} - \frac{nb_0^2}{r^2(nr^2+b_0^2)} + \frac{b_0^4}{r^4(nr^2+b_0^2)} + \frac{2n(r^2-b_0^2)(nr^2-3b_0^2)}{r^2(nr^2+b_0^2)^2}$



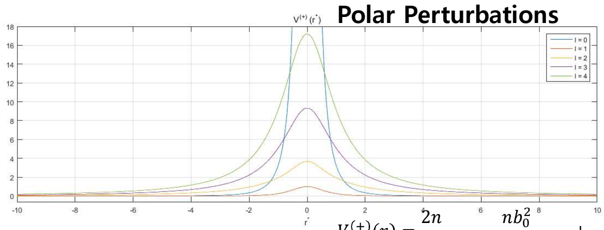
GPs : Linearized Theory

Axial Perturbations



$$V^{(-)}(r) = \frac{l(l+1)}{r^2} - \frac{3b_0^2}{r^4}$$

Polar Perturbations



$$V^{(+)}(r) = \frac{2n}{r^2} - \frac{nb_0^2}{r^2(nr^2+b_0^2)} + \frac{b_0^4}{r^4(nr^2+b_0^2)} + \frac{2n(r^2-b_0^2)(nr^2-3b_0^2)}{r^2(nr^2+b_0^2)^2}$$

GPs : Linearized Theory

Axial Perturbations

r	$l=2$	$l=3$	$l=4$	r	$l=2$	$l=3$	$l=4$
1	3.00000	9.00000	17.0000	2.4	0.95124	1.99291	3.3818
1.1	2.90964	7.86832	14.47989	2.6	0.82193	1.7095	2.89293
1.2	2.71991	6.88657	12.44213	2.8	0.7165	1.4818	2.50221
1.3	2.49991	6.05021	10.78394	3	0.62963	1.2963	2.18519
1.4	2.2803	5.34152	9.42316	3.5	0.4698	0.9596	1.61266
1.5	2.07407	4.74074	8.2963	4	0.36328	0.73828	1.23828
1.6	1.88599	4.22974	7.35474	5	0.2352	0.4752	0.7952
1.7	1.71693	3.79306	6.56122	6	0.16435	0.33102	0.55324
1.8	1.56607	3.41792	5.88706	7	0.1212	0.24365	0.40691
1.9	1.43185	3.0939	5.30997	8	0.09302	0.18677	0.31177
2	1.3125	2.8125	4.8125	9	0.07362	0.14769	0.24646
2.2	1.1116	2.35127	4.00417	10	0.0597	0.1197	0.1997

Polar Perturbations

r	$l=2$	$l=3$	$l=4$	r	$l=2$	$l=3$	$l=4$
1	3.66667	9.33333	17.2000	2.4	0.84879	1.94808	3.3562
1.1	2.98777	7.88171	14.48226	2.6	0.74291	1.67537	2.87352
1.2	2.53437	6.77067	12.36793	2.8	0.65493	1.45548	2.48729
1.3	2.20896	5.8903	10.68545	3	0.58114	1.27574	2.17356
1.4	1.96079	5.17548	9.32278	3.5	0.44185	0.94793	1.60609
1.5	1.76227	4.58459	8.20307	4	0.34619	0.73122	1.23432
1.6	1.59775	4.08936	7.27171	5	0.22785	0.4722	0.79352
1.7	1.45787	3.66967	6.48879	6	0.16071	0.32954	0.55242
1.8	1.33678	3.31071	5.8245	7	0.1192	0.24284	0.40646
1.9	1.23059	3.00124	5.25618	8	0.09184	0.18629	0.3115
2	1.13657	2.73257	4.7663	9	0.07287	0.14739	0.24629
2.2	0.97761	2.29168	3.96996	10	0.05921	0.1195	0.19959

GPs and Exotic Matter

- Traversable wormholes are not vacuum ⇒ perturbed exotic matter
- Einstein equations $\delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu}$
- Energy conservation $\delta\Gamma^{\mu}_{\nu\delta} T^{\delta\nu} + \delta\Gamma^{\nu}_{\nu\delta} T^{\delta\mu} + \partial_\nu \delta T^{\mu\nu} + \Gamma^{\mu}_{\nu\delta} \delta T^{\delta\nu} + \Gamma^{\nu}_{\nu\delta} \delta T^{\delta\mu} = 0$
- Perturbed stress-energy tensor

$$\delta T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \delta\rho & 0 & 0 & 0 \\ 0 & -\delta\tau & 0 & 0 \\ 0 & 0 & \delta p & 0 \\ 0 & 0 & 0 & \delta p \end{pmatrix}$$

- Axial perturbations stay the same
- Polar perturbations changes
  - $\delta G_{\mathrm{tr}} = \delta G_{t\theta} = \delta G_{r\theta} = 0$
  - $\delta G_{\mu\mu} \neq 0$

GPs and Exotic Matter

$\delta G_{tr} = 0$	$\frac{\partial L}{\partial r} + \frac{2}{r}L = -\frac{\partial X}{\partial r} - \frac{1}{r}X$
$\delta G_{t\theta} = 0$	$T - V + L = 0$
$\delta G_{r\theta} = 0$	$\frac{\partial N}{\partial r} - \frac{1}{r}N = \frac{\partial L}{\partial r} + \frac{1}{r}L$

$$\delta G_{tt} = 2\left[-\frac{r^2 - 2b_0^2}{r^4}L - \frac{b_0^2}{r^4}N\right]P_l$$

$$\delta G_{rr} = \frac{2r^2}{r^2 - b_0^2}\left[\frac{\partial^2}{\partial t^2}(L + X) - \frac{(r^2 - b_0^2)}{r^3}\frac{\partial X}{\partial r} + \frac{(r^2 - b_0^2)}{r^4}(N + L) - \frac{1}{r^2}(2L + X)\right]P_l$$

$$\delta G_{\theta\theta} = r^2\left[\left\{-\frac{\partial^2 V}{\partial t^2} + \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} + \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} + \frac{1}{r^2}(L + N) - \frac{2b_0^2}{r^4}V - \frac{4b_0^2}{r^4}(L + X)\right\}P_l + \left\{-\frac{\partial^2 V}{\partial t^2} + \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} + \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} + \frac{1}{r^2}(L + N) - \frac{2b_0^2}{r^4}V\right\}\cot\theta\frac{\partial P_l}{\partial \theta}\right]$$

$$\delta G_{\phi\phi} = r^2\sin^2\theta\left[\left\{-\frac{\partial^2 V}{\partial t^2} + \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} + \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} + \frac{1}{r^2}(L + N) - \frac{2b_0^2}{r^4}V - \frac{4b_0^2}{r^4}(L + X)\right\}P_l + \left\{-\frac{\partial^2 V}{\partial t^2} + \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} + \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} + \frac{1}{r^2}(L + N) - \frac{2b_0^2}{r^4}V\right\}\frac{\partial^2 P_l}{\partial \theta^2}\right]$$

GPs and Exotic Matter

$$\frac{\partial^2 V}{\partial t^2} - \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} - \frac{(L + N)}{r^2} - \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} + \frac{2b_0^2}{r^4}V = 0$$

$$\delta G_{tt} = 2\left[-\frac{(r^2 - 2b_0^2)}{r^4}L - \frac{b_0^2}{r^4}N\right]P_l$$

$$\delta G_{rr} = \frac{2r^2}{(r^2 - b_0^2)}\left[\frac{\partial^2}{\partial t^2}(L + X) - \frac{(r^2 - b_0^2)}{r^3}\frac{\partial X}{\partial r} + \frac{(r^2 - b_0^2)}{r^4}N - \frac{(r^2 + b_0^2)}{r^4}L - \frac{X}{r^2}\right]P_l$$

$$\delta G_{\theta\theta} = -\frac{4b_0^2}{r^2}(L + X)P_l$$

$$\delta G_{\phi\phi} = -\frac{4b_0^2}{r^2}\sin^2\theta(L + X)P_l$$

In orthogonal basis
$\delta G_{\hat{\theta}\hat{\theta}} = \delta G_{\hat{\phi}\hat{\phi}} = \delta p$

- Energy conservation law

$$\delta\rho = \frac{2b_0^2}{r^4}(N + L)P_l$$

$$\partial_r(\delta\tau) + \frac{2}{r}\delta\tau = \frac{2b_0^2}{r^4}\left[2\frac{dX}{dr} - \frac{1}{r}(N - 3L + 2X)\right]P_l$$

$$\delta p = \frac{4b_0^2}{r^4}X\, P_l$$

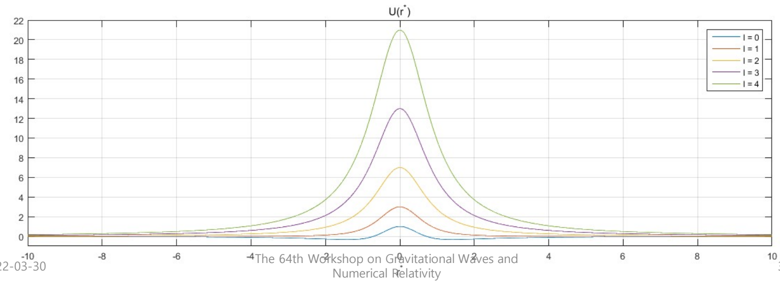
$$L + N = -2X$$

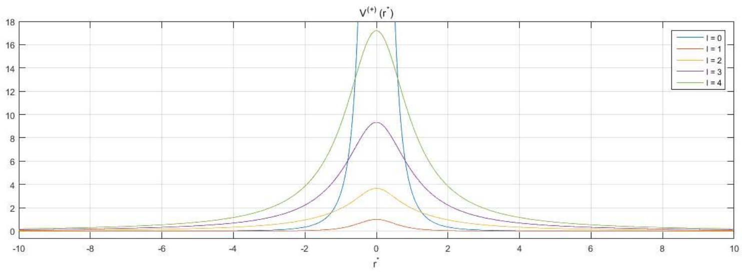
GPs and Exotic Matter

$$\frac{\partial^2 V}{\partial t^2} - \frac{(r^2 - b_0^2)}{r^2}\frac{\partial^2 V}{\partial r^2} - \frac{(2r^2 - b_0^2)}{r^3}\frac{\partial V}{\partial r} - \frac{3n}{r^2}V + \frac{2b_0^2}{r^4}V = 0$$

- Assumption 1 : time dependence $\propto e^{i\omega' t}$
- Assumption 2 : $V(r) = \frac{R(r)}{r}$

$$\frac{d^2R}{dr^{*2}} + \left(\omega'^2 - U(r)\right)R = 0, \qquad U(r) = \frac{(l-1)(l+2)}{r^2} + \frac{3b_0^2}{r^4}.$$





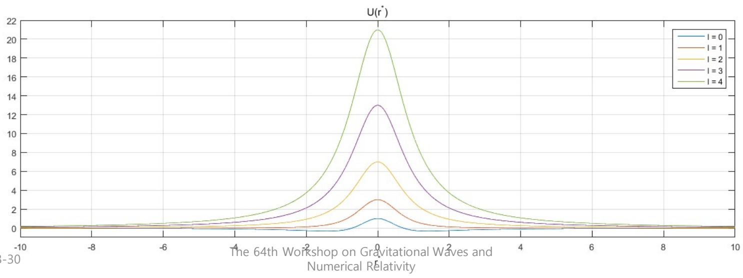
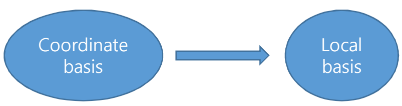


Table of Contents

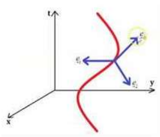
Newman-Penrose Formalism

- Tetrad Formalism

: useful to generalize the choice of basis.



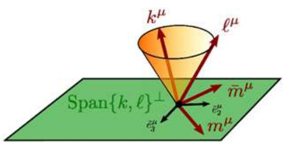
- Build a null tetrad from an orthonormal tetrad



- Null vectors $(l^{\mu}, k^{\mu}, m^{\mu}, m^{*\mu})$
  - two real and a complex conjugate pair.
- The propagation of radiation in curved spacetime.

Newman-Penrose Formalism

- Null vectors $(l^\mu, k^\mu, m^\mu, m^{*\mu})$
- l^μ & k^μ
  - real quantities along the light cone
  - oriented toward the future



- m^μ & $m^{*\mu}$
- Orthogonality conditions

$$l \cdot m = l \cdot m^* = k \cdot m = k \cdot m^* = 0$$
$$l \cdot l = k \cdot k = m \cdot m = m^* \cdot m^* = 0$$
$$l \cdot k = -m \cdot m^* = 1$$

Newman-Penrose Formalism

- Metric tensor $g_{\mu\nu}$

$$g_{\mu\nu} = l_\mu k_\nu + k_\mu l_\nu - m_\mu m_\nu^* - m_\mu^* m_\nu$$

- Directional Operators

$$D = l^\mu \partial_\mu$$
$$\Delta = k^\mu \partial_\mu$$
$$\delta = m^\mu \partial_\mu$$

- Twelve complex spin coefficients
- Five Weyl tensors $\Psi_0, \dots, \Psi_4$
- Ten Ricci Tensors
- Ψ_4

: encoding the outgoing-gravitational radiation of an asymptotically flat system.

Newman-Penrose Formalism

- Teukolsky S. A. 1972 *Phys. Rev. Lett.* **29** 1114

➢ linearized perturbation equation using NP formalism (type D)

$$\bar{l} = l + l^{(1)}$$

$$\bar{k} = k + k^{(1)}$$

^{A} : the background spacetime

^{B} : the perturbed spacetime

$$[(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) - (\delta + \pi^* - \alpha^* - 3\beta - 3\tau)(\delta^* + \pi - 4\alpha) - 3\Psi_2^A]\Psi_0^{(1)} = 4\pi T_0$$

$$\left[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D + 4\epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)\left(\delta \quad - \pi + 3\beta\right) - 3\Psi_2^A\right]\Psi_4^{(1)} = 4\pi T_4$$

Newman-Penrose Formalism

- Teukolsky equations for the perturbed Weyl scalar Ψ_4 with source terms T_4

$$\left[\left(\Delta + \eta(4\mu_s + \mu_s^* + 3\gamma_s - \gamma_s^*)\right)\left(D - \eta(\rho_s - 4\epsilon_s)\right) - \left(\delta^* + \eta(3\alpha_s + \beta_s^* + 4\pi_s - \tau_s^*)\right)\left(\delta \quad + \eta(4\beta_s - \tau_s)\right) - 3\eta\Psi_2 \right]\Psi_4^{(1)} = \eta\frac{K}{2}T_4$$

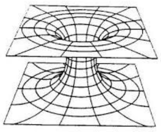
$K = 8\pi$, $\eta = -1$ for $(-,+,+,+)$

- Source term, T_4

$$T_4 = \hat{T}^{kk}T_{kk} + \hat{T}^{km^*}T_{km^*} + \hat{T}^{m^*m^*}T_{m^*m^*}$$

The Morris-Thorne Wormhole and the Newman-Penrose Formalism

- The Morris-Thorne Wormhole



$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega^2$$

$\Phi(r)$: red-shift function

$b(r)$: shape function

- The null vectors (the radial null-geodesics)

$\boldsymbol{l \cdot k = \eta = -1}$

$$l^\mu = \frac{1}{2}\left(e^{-2\Phi}, e^{-\Phi}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}, 0,0\right)$$

$$k^\mu = \left(1, -e^{\Phi}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}, 0, 0\right)$$

$$m^\mu = \frac{1}{\sqrt{2}r}(0, 0, 1, i\csc\theta)$$

The Morris-Thorne Wormhole and the Newman-Penrose Formalism

- The directional derivatives (Operators)

$$D = l^\mu \nabla_\mu$$

$$= \frac{1}{2}e^{-2\Phi}\left[\frac{\partial}{\partial t} + e^{\Phi}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}\right]$$

$$\Delta = k^\mu \nabla_\mu = \frac{\partial}{\partial t} - e^{\Phi}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}$$

$$\delta = m^\mu \nabla_\mu = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

- Spin coefficients (non-vanishing)

$$\rho_s = \frac{r-b}{2r^2}e^{-\Phi}\left(1-\frac{b}{r}\right)^{-\frac{1}{2}}$$

$$\mu_s = \frac{e^{\Phi}}{r}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}$$

$$\gamma_s = -\Phi' e^{\Phi}\left(1-\frac{b}{r}\right)^{\frac{1}{2}}$$

$$\beta_s = -\frac{1}{2\sqrt{2}r}\frac{\cos\theta}{\sin\theta} = -\alpha_s$$

$$\kappa_s = \tau_s = \sigma_s = \pi_s = \nu_s = \lambda_s = \epsilon_s = 0$$

Type D spacetime

The Morris-Thorne Wormhole and the Newman-Penrose Formalism

$$\Big[\big(\Delta + \eta(4\mu_s + \mu_s^* + 3\gamma_s - \gamma_s^*)\big)\big(D - \eta(\rho_s - 4\epsilon_s)\big) - \big(\delta^* + \eta(3\alpha_s + \beta_s^* + 4\pi_s - \tau_s^*)\big)\Big(\delta \quad + \eta(4\beta_s - \tau_s)\Big) - 3\eta\Psi_2 \Big]\Psi_4^{(1)} = \eta\frac{K}{2}T_4$$

- Perturbation equation

$$\left[\Box_{tr}^{\Psi} + \Box_{\theta\phi}\right]\Psi_4^{(1)} = Kr^2T_4$$

$$\Box_{tr}^{\Psi} = -r^2e^{-2\Phi}\frac{\partial^2}{\partial t^2} + r(r-b)\frac{\partial^2}{\partial r^2} - 4e^{-\Phi}(r^2 - br)^{\frac{1}{2}}(r\Phi' - 1)\frac{\partial}{\partial t} - \left[\frac{b'r-b}{2} + 3(\Phi'r - 2)(r-b)\right]\frac{\partial}{\partial r} - 3(3\Phi'r - 1)\left(1 - \frac{b}{r}\right) + (1 - b') + \frac{7(b'r-b)}{2r}$$

$$\Box_{\theta\phi} = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - 4i\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} - \frac{2}{\sin^2\theta}(1 + \cos^2\theta)$$

The Morris-Thorne Wormhole and the Newman-Penrose Formalism

- Perturbation equation

$$\left[\Box_{tr}^{\Psi} + \Box_{\theta\phi}\right]\Psi_4^{(1)} = Kr^2T_4$$

$$\hat{T}^{kk} = -\frac{1}{2r^2}\bar{\partial}_{-1}\bar{\partial}_0,$$

$$T_4 = \hat{T}^{kk}T_{kk} + \hat{T}^{km^*}T_{km^*} + \hat{T}^{m^*m^*}T_{m^*m^*}$$

$$\hat{T}^{km^*} = -\frac{1}{\sqrt{2}}\left[\frac{2}{r}\left[\frac{\partial}{\partial t} - e^{\Phi}\left(1 - \frac{b}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}\right] - \frac{e^{\Phi}}{r}\left(1 - \frac{b}{r}\right)^{\frac{1}{2}}\left(\frac{5}{r} - 4\Phi'\right)\right]\bar{\partial}_{-1}$$

$$\hat{T}^{m^*m^*} = -\left[\frac{\partial^2}{\partial t^2} - 2e^{\Phi}\left(1 - \frac{b}{r}\right)^{\frac{1}{2}}\frac{\partial^2}{\partial t\partial r} + e^{2\Phi}\left(1 - \frac{b}{r}\right)\frac{\partial^2}{\partial r^2} - 2e^{\Phi}\left(1 - \frac{b}{r}\right)^{\frac{1}{2}}\left(\frac{3}{r} - \Phi'\right)\frac{\partial}{\partial t} + e^{2\Phi}\left[\left(1 - \frac{b}{r}\right)\left(\frac{6}{r} - \Phi'\right) - \frac{(b'r-b)}{2r^2}\right]\frac{\partial}{\partial r} + e^{2\Phi}\left[\left(1 - \frac{b}{r}\right)\left(\frac{4}{r^2} - \frac{\Phi'}{r}\right) - \frac{(b'r-b)}{2r^3}\right]\right]$$

Spin operators

$$\eth_s \equiv \eth_0 + s \cot\theta$$
$$\bar{\eth}_s \equiv \bar{\eth}_0 - s \cot\theta$$

- On the spin-weighted spherical harmonics $Y_s^{l,m}(\theta, \phi)$

$$\eth_s Y_s^{l,m} = \sqrt{(l-s)(l+s+1)}\,Y_{s+1}^{l,m}$$
$$\bar{\eth}_s Y_s^{l,m} = -\sqrt{(l+s)(l-s+1)}\,Y_{s-1}^{l,m}$$

$\Phi(r) = 0, b(r) = \frac{b_0^2}{r}$ 인 경우,

- Perturbation equation

$$\left[\Box_{tr}^{\Psi} + \Box_{\theta\phi}\right]\Psi_4^{(1)} = Kr^2 T_4$$

$$\Box_{tr}^{\Psi} = -r^2 \frac{\partial^2}{\partial t^2} + (r^2 - b_0^2)\frac{\partial^2}{\partial r^2} - 4(r^2 - b_0^2)^{\frac{1}{2}}\frac{\partial}{\partial t} + r\left(6 - 5\frac{b_0^2}{r^2}\right)\frac{\partial}{\partial r} + \left(4 - 9\frac{b_0^2}{r^2}\right)$$

$$\Box_{\theta\phi} = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - 4i\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} - \frac{2}{\sin^2\theta}(1 + \cos^2\theta)$$

- $r\Psi_4^{(1)} = \Phi_4^{(1)}$

$$\left[\Box_{tr}^{\Phi_4^{(1)}} + \Box_{\theta\phi}\right]\Phi_4^{(1)} = Kr^2 T_4$$

$$\Box_{tr}^{\Phi_4^{(1)}} = -r^2 \frac{\partial^2}{\partial t^2} + (r^2 - b_0^2)\frac{\partial^2}{\partial r^2} - 4(r^2 - b_0^2)^{\frac{1}{2}}\frac{\partial}{\partial t} + r\left(4 - \frac{3b_0^2}{r^2}\right)\frac{\partial}{\partial r} - 6\frac{b_0^2}{r^2}$$

$$\Box_{\theta\phi} = \bar{\eth}_{-1}\eth_{-2}$$

The Morris-Thorne Wormhole and the Newman-Penrose Formalism

$$\Phi_4^{(1)} = \sum_{l,m} R_{l,m}(t,r) Y_{-2}^{lm}(\theta,\phi)$$

$$\square_{tr}^{\Phi_4^{(1)}} R_{l,m} - (l-1)(l+2)R_{l,m} = Kr^3 T_4$$

- Proper radial distance, $r^*(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1-\frac{b_\pm(r)}{r}}}$

$$\square_{tr^*}^{\Phi_4^{(1)}} R_{l,m} - (l-1)(l+2)R_{l,m} = Kr(r^*)^3 T_4$$

$$\square_{tr^*}^{\Phi_4^{(1)}} = -(r^{*2}+b_0^2)\frac{\partial^2}{\partial t^2} + (r^{*2}+b_0^2)\frac{\partial^2}{\partial r^{*2}} + 4r^*\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r^*}\right) - \frac{6b_0^2}{(r^{*2}+b_0^2)}$$

Fourier decomposition

$$R_{l,m}(t,r^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\xi_{lm}(\omega', r^*)d\omega$$

$$\frac{\partial^2 \xi_{lm}^{\omega}}{\partial r^{*2}} + \frac{4r^*}{(r^{*2}+b_0^2)}\frac{\partial \xi_{lm}^{\omega}}{\partial r^*} + \left(\omega^2 - \frac{4ir^*\omega + (l-1)(l+2)}{(r^{*2}+b_0^2)} - \frac{6b_0^2}{(r^{*2}+b_0^2)^2}\right)\xi_{lm}^{\omega} = 0$$

- $\xi_{lm}^{\omega} \equiv \frac{\chi(r^*)}{(r^{*2}+b_0^2)}$

$$\frac{d^2 \chi_{lm}^{\omega}}{dr^{*2}} + \left(\omega^2 - \frac{4ir^*\omega + (l-1)(l+2)}{(r^{*2}+b_0^2)} - \frac{6b_0^2}{(r^{*2}+b_0^2)^2}\right)\chi_{lm}^{\omega} = 0$$

- Far from the throat of the wormhole $(r^* \to \infty)$

$$\frac{d^2 \chi_{lm}^{\omega}}{dr^{*2}} + \left(\omega^2 - \frac{4i\omega}{r^*}\right)\chi_{lm}^{\omega} \approx 0$$

- The asymptotic solutions $\chi_{lm}^{\omega} \sim r^{*\mp 2} e^{\mp i\omega\ ^*}$

+	Outgoing waves
-	Ingoing waves

결과

- Outgoing waves

$$\Psi_4^{(1)} \sim \frac{e^{i\omega r^*}}{r^*}$$

- Ingoing waves

$$\Psi_4^{(1)} \sim \frac{e^{-i\omega r^*}}{r^{*5}}$$

- One-dimensional Schrödinger-like equation

- Potential

$$V(r^*) = \frac{4i\omega r^*}{(r^{*2}+b_0^2)} + \frac{(l-1)(l+2)}{(r^{*2}+b_0^2)} + \frac{6b_0^2}{(r^{*2}+b_0^2)^2}$$

Conclusion

- Traversable wormholes oscillate when perturbed.
- The oscillations have unique frequencies: quasi normal modes
- An oscillating traversable wormhole with mass in the range

$$10^2 M_\odot \le M \le 10^4 M_\odot$$

can be detected by the ground based gravitational interferometer like advanced LIGO and Virgo.

$$10^6 M_\odot \le M \le 10^9 M_\odot$$

can be detected by the space based interferometer like LISA.

Latest Studies related to Today's Topic

- J. C. D. Aguila and T. Matos, Phys. Rev. D 103(2021) 084033
  - Gravitational perturbations in the Newman-Penrose formalism: Application to wormholes

Thank you for listening!