2022.10.27

Workshop on Gravitational Wave and Numerical Relativity

Gravitational Lensing of Gravitational Waves

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- Why are we interested in the wave optics of GW ?

II. Wave optics Lensing Formalism

- Derivation of wave optics amplification factor

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- Brief Review of recent studies with wave optics of GW

IV. Probing low mass dark matter halo

- Introduce my work on gravitational lensing of gravitational waves
- "Small-scale shear: peeling off diffuse subhalos with gravitational waves"

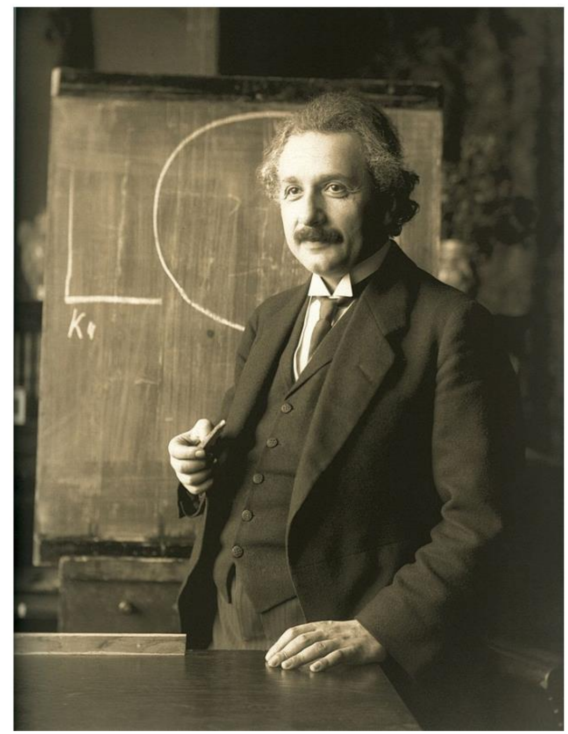
  Han Gil Choi, Chanung Park and Sunghoon Jung, arXiv : 2103.08618[astro-ph.CO]

I. Introduction

107 years ago..
Special Relativity + Equivalence principle
= General Relativity

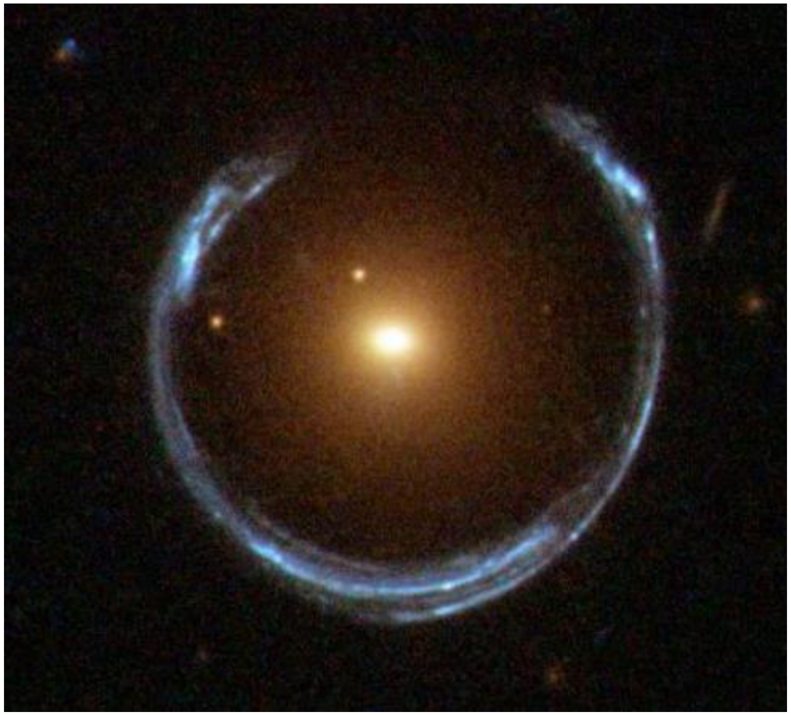
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

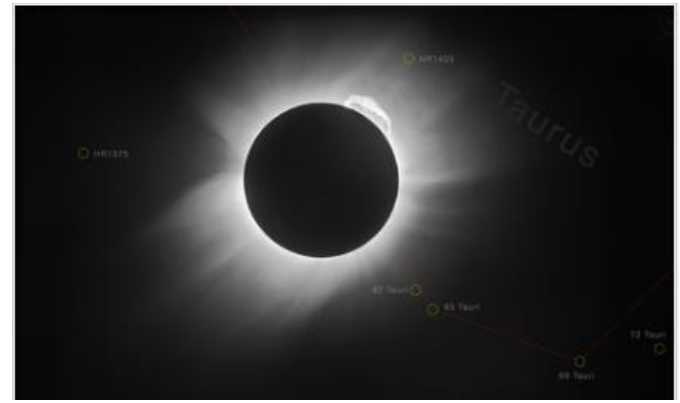
Mass(energy) and Space-Time
are coupled!



Einstein in 1921

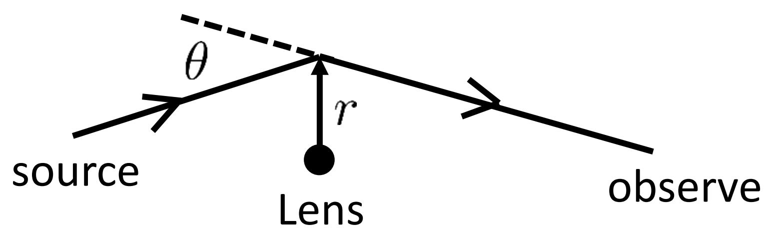
I. Introduction





1919 Eddington’s experiment

$$\text{Deflection angle}:\ \theta = \frac{4GM}{c^2 r}$$

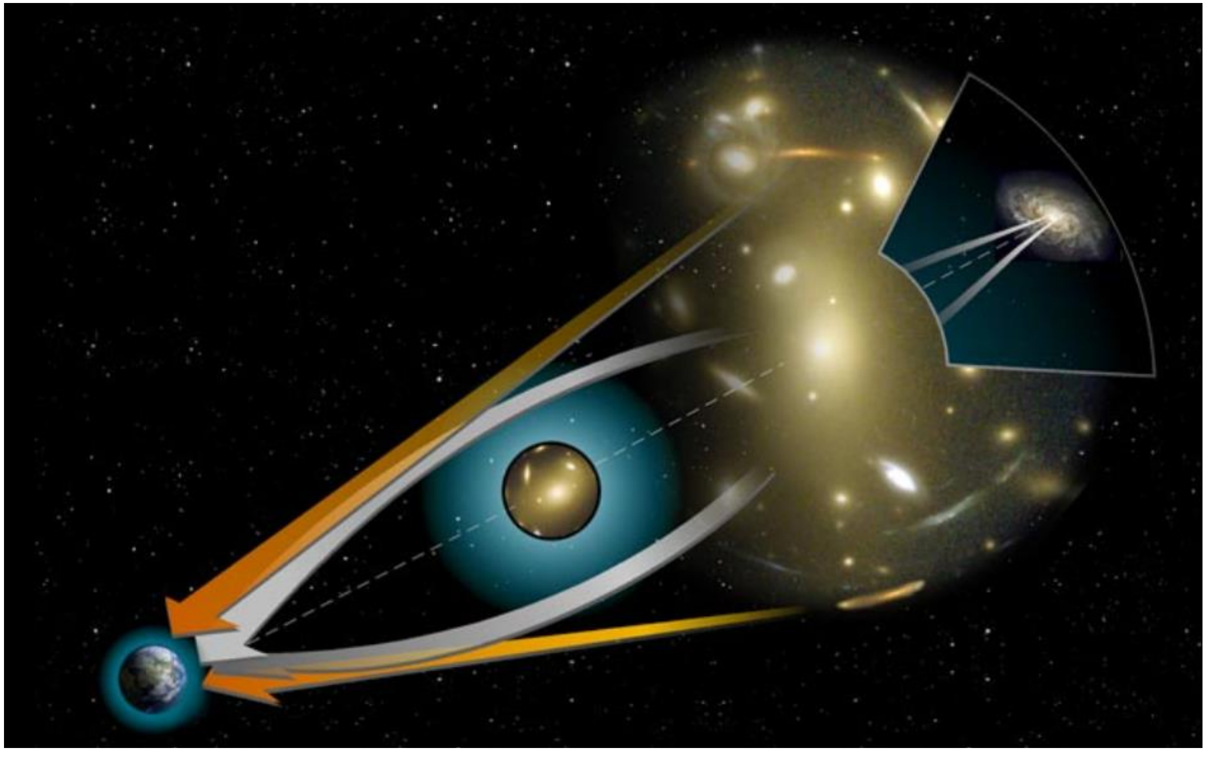


GR : Mass can deflect light propagation!

I. Introduction

Two types of gravitational lensing(GL)

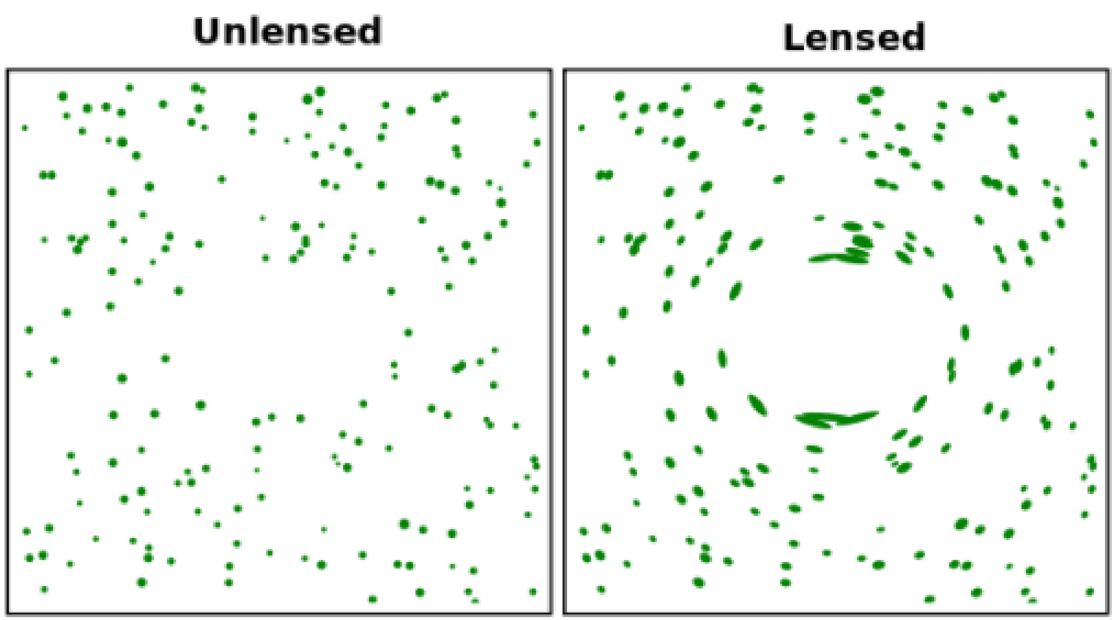
- Strong GL
  - Lensing by compact object (cf) galaxy)
  - Multiple lensed Images (cf. Einstein ring)



I. Introduction

Two types of gravitational lensing(GL)

- Weak GL
  - Lensing by diffuse object or **far from** compact object
  - No multiple image
  - Only Image distortion



I. Introduction

Modern Science with Gravitational lensing

- Mass profile of galaxies
- Dark matter distribution (e. g. bullet cluster)
- Exo-planet search (Microlensing)
- Cosmology
  - Shear correlation function
  - Hubble constant measurement (lensed quasars, time delay)
- and etc…

I. Introduction

What about Gravitational waves?

- Science of Gravitational lensing of GW will be very different

	Gravitational wave	Electromagnetic
Measurement	Amplitude, Phase	Intensity
Imaging	No	Yes
Frequency band	10 nHz ~ 10 kHz (1 pc ~ 30 km)	> 1 GHz (radio) (> 0.3 m)
Number of sources	Rare (<1000 1/yr/Gpc^3)	Many (e. g. SNIa) (>10000 1/yr/Gpc^3)
Source size	< 1 AU	> 1 AU

*Imaging requires dense measurement within an aperture $d \gg \lambda$

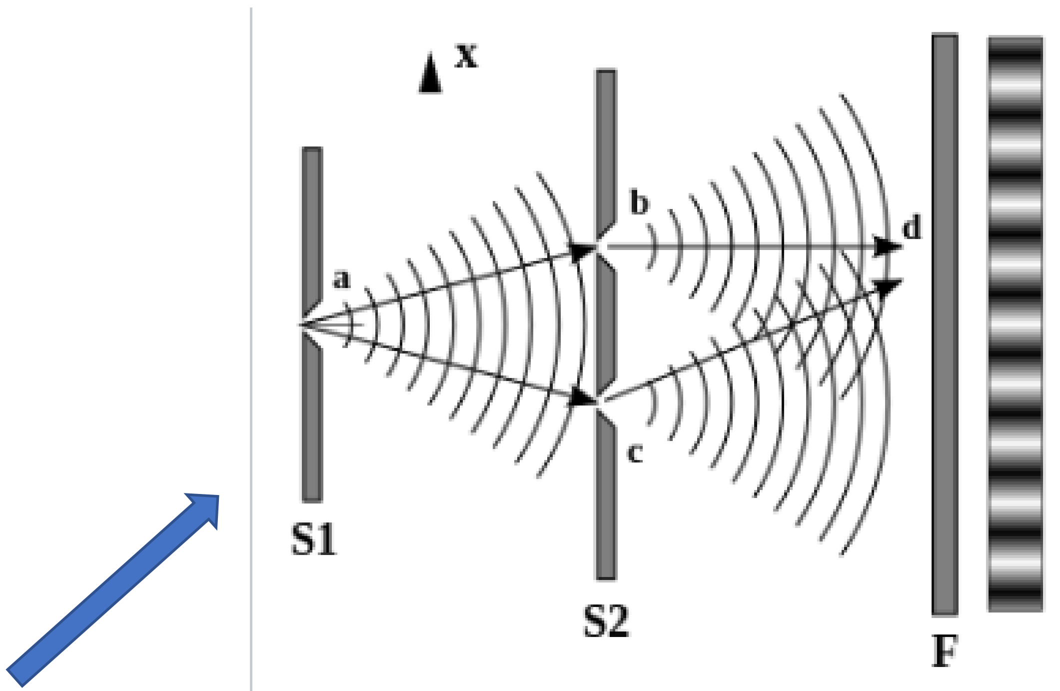
*Interferometer network : $d \sim 1000\ km$ with only a few observing points

I. Introduction

Low frequency , small source size

⇒ Principles of **Wave** propagation is important !

*why small source size ?



If light source is too big, S1 is necessary

II. Wave optics lensing formalism

Basic principles

- Curved space-time $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- Field equation & Linear perturbation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$
$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$
$$\Rightarrow \quad \Box_0 h^{TT}_{\mu\nu} = 0$$

- Weakly curved background

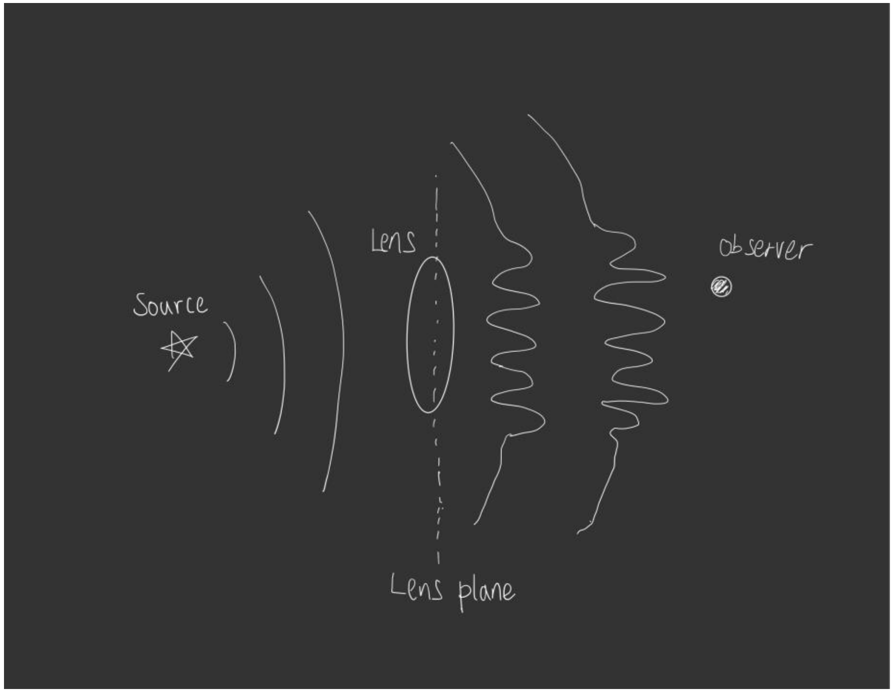
$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu = -(1+2U)dt^2 + (1-2U)d\mathbf{x}^2$$
$$\nabla^2 U = 4\pi\rho$$

- Negligible polarization change $h^{TT}_{\mu\nu} \simeq \phi e_{\mu\nu}$
- Leading order in U $(-\partial_t^2 + \nabla^2)\phi(t, \mathbf{x}) = -4U(\mathbf{x})\partial_t^2 \phi(t, \mathbf{x})$

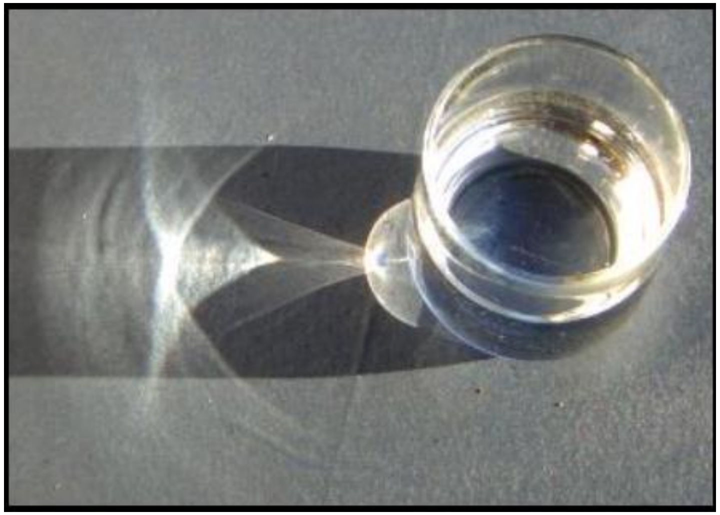
II. Wave optics lensing formalism

How to solve ? $(-\partial_t^2 + \nabla^2)\phi(t,\mathbf{x}) = -4U(\mathbf{x})\partial_t^2\phi(t,\mathbf{x})$

1. Thin lens approximation + Kirchhoff integral theorem
   - Reproduce geometric optics
   - Can deals with '**Caustics**'
2. Born's approximation
   - Easy to calculate (both analytically, numerically)
3. Partial wave expansion? (particle physics)



Example of caustics



II. Wave optics lensing formalism

Solving with Kirchoff integral theorem

- Frequency domain $(\nabla^2 + w^2)\tilde{\phi}(w, \mathbf{x}) = 4w^2 U(\mathbf{x})\tilde{\phi}(w, \mathbf{x})$

- Thin lens approximation, **outside** the lens plane $(\nabla^2 + w^2)\tilde{\phi}(w, \mathbf{x}) \simeq 0$

- Green's theorem

$$\int_V dx^3 \tilde{\phi}\nabla^2\tilde{\phi}' - \tilde{\phi}'\nabla^2\tilde{\phi} = -\int_S dS\tilde{\phi}\frac{\partial\tilde{\phi}'}{\partial n} - \tilde{\phi}'\frac{\partial\tilde{\phi}}{\partial n}$$

We can choose any ϕ' satisfying $(\nabla^2 + w^2)\tilde{\phi}'(w, \mathbf{x}) \simeq 0$

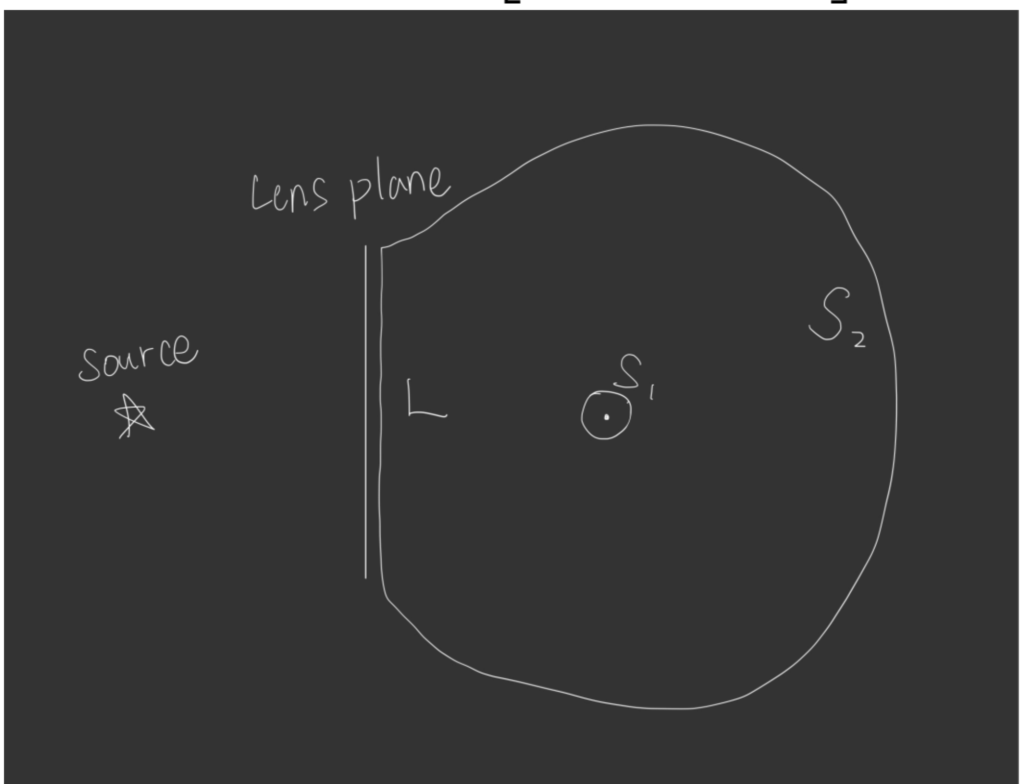
$$\Rightarrow \int_S dS\tilde{\phi}\frac{\partial\tilde{\phi}'}{\partial n} - \tilde{\phi}'\frac{\partial\tilde{\phi}}{\partial n} = 0$$

II. Wave optics lensing formalism

Solving with Kirchoff integral theorem

- We choose 'S' by

$$\left(\int_{S_1} + \int_{S_2} + \int_{L}\right) dS \left[\tilde{\phi}\frac{\partial \tilde{\phi}'}{\partial n} - \tilde{\phi}'\frac{\partial \tilde{\phi}}{\partial n}\right] = 0$$



II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

- We choose $\tilde{\phi}'(\mathbf{x}) = \dfrac{e^{iw|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}$ centered at S_1, $\mathbf{x}_0$

$$-\int_{S_1} dS \left[\tilde{\phi}\frac{\partial\tilde{\phi}'}{\partial n} - \tilde{\phi}'\frac{\partial\tilde{\phi}}{\partial n}\right] \simeq -\int d\Omega\, \epsilon^2 \left[\tilde{\phi}\frac{e^{iw\epsilon}}{\epsilon}\left(iw - \frac{1}{\epsilon}\right) - \frac{e^{iw\epsilon}}{\epsilon}\frac{\partial\tilde{\phi}}{\partial r}\right]$$

$$= 4\pi\tilde{\phi}(\mathbf{x}_0)\,,\, \epsilon \to 0$$

$$\int_{S_2} dS \left[\tilde{\phi}\frac{\partial\tilde{\phi}'}{\partial n} - \tilde{\phi}'\frac{\partial\tilde{\phi}}{\partial n}\right] \simeq 0,\ S_2 \to \infty$$

$$\Rightarrow \phi(\mathbf{x}_0) = \frac{1}{4\pi}\int_L dS \left[\tilde{\phi}\frac{\partial}{\partial n}\left(\frac{e^{iw|\mathbf{x}'-\mathbf{x}_0|}}{|\mathbf{x}'-\mathbf{x}_0|}\right) - \left(\frac{e^{iw|\mathbf{x}'-\mathbf{x}_0|}}{|\mathbf{x}'-\mathbf{x}_0|}\right)\frac{\partial\tilde{\phi}}{\partial n}\right]$$

II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

- We need to find $\tilde{\phi}(\mathbf{x}')$ (right behind the lens plane)

- Eikonal approximation to wave propagation (nearly planewave)

$$\tilde{\phi}(\mathbf{x}) = A(\mathbf{x})e^{iS(\mathbf{x})} \quad \text{with} \quad |\nabla S| \gg |A^{-1}\nabla A|,\ |\nabla^2 S|^{1/2}$$

$$\partial_i S \partial^i S = w^2(1-4U)$$

- Assuming point source $\phi(\mathbf{x}') = A_0 e^{iw|\mathbf{x}'|}/|\mathbf{x}'|$

$$A(\mathbf{x}') = A_0/|\mathbf{x}'|$$

$$S(\mathbf{x}') = w\int_{\mathrm{source}}^{\mathbf{x}'} dl(1-2U) \simeq w(|\mathbf{x}'| - \psi(\mathbf{x}'))$$

2d projected potential $\psi(\mathbf{x}') \equiv 2\int_{-\infty}^{\infty} dl\, U(\mathbf{x}', l)$

II. Wave optics lensing formalism

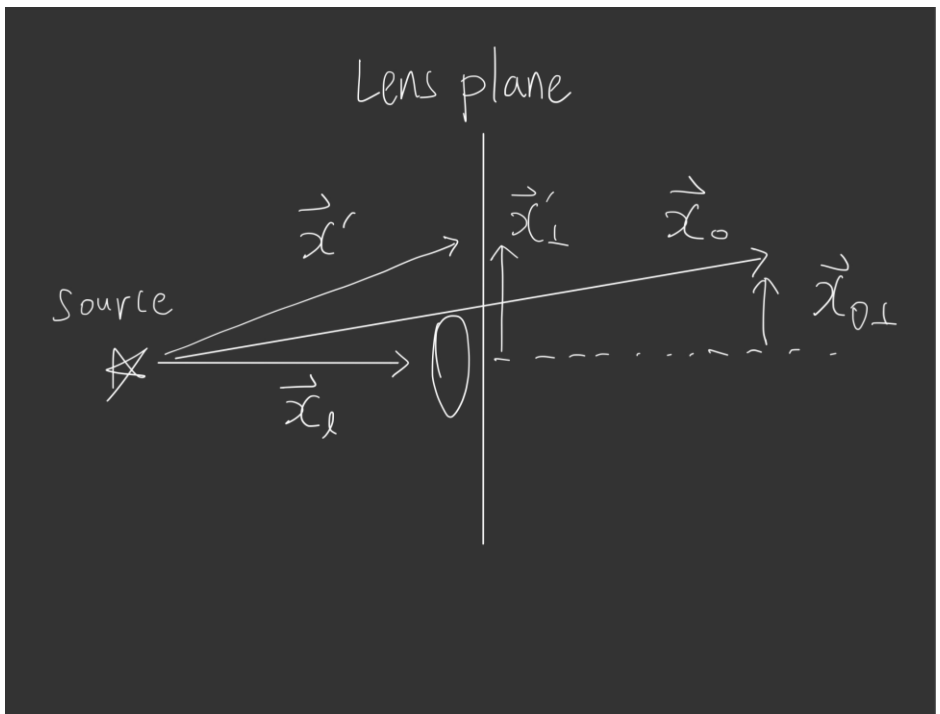
Solving with Kirchhoff integral theorem

- We apply Paraxial approximation

$$\mathbf{x}' = \mathbf{x}_l + \mathbf{x}'_\perp \qquad \text{with} \qquad |\mathbf{x}_l|,\ |\mathbf{x}_s| \gg |\mathbf{x}'_\perp|,\ |\mathbf{x}_{0\perp}|$$

$$\mathbf{x}_0 = \mathbf{x}_s + \mathbf{x}_{0\perp}$$

Ex) $|\mathbf{x}'| \simeq d_l + \dfrac{|\mathbf{x}'_\perp|^2}{2d_l} \qquad |\mathbf{x}_0| \simeq d_s + \dfrac{|\mathbf{x}_{0\perp}|^2}{2d_s}$



II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

- Taking leading orders, we have

$$\phi(\mathbf{x}_0) \simeq A_0 \frac{e^{iw|\mathbf{x}_0|}}{|\mathbf{x}_0|} \left[\frac{w}{2\pi i d_{\rm eff}} \int dx_\perp'^2 \exp\left(iwt_d(\mathbf{x}_\perp')\right) \right]$$

$$d_{\rm eff} = \frac{d_l(d_s - d_l)}{d_s} \qquad d_{\rm eff} \to d_l \; (d_s \to \infty)$$

$$t_d(\mathbf{x}_\perp') = \frac{1}{2d_{\rm eff}} |\mathbf{x}_\perp' - \frac{d_l}{d_s}\mathbf{x}_{0\perp}|^2 - \psi(\mathbf{x}_\perp')$$

Geo. Grav.

- Lensing amplification factor $F(w, \mathbf{x}_0) \equiv \frac{w}{2\pi i d_{\rm eff}} \int dx_\perp'^2 \exp\left(iwt_d(\mathbf{x}_\perp')\right)$

F = 1 when $\psi = 0$

II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

How to evaluate the integral?

$$F(w, \mathbf{x}_0) \equiv \frac{w}{2\pi i d_{\text{eff}}} \int dx'^2_\perp \exp\left(iwt_d(\mathbf{x}'_\perp)\right)$$

- Exact solution allowed only for special cases

1. Frequency domain methods
   - Stationary phase approximation, only for geometric optics limit $f \to \infty$
   - Levin's oscillatory integral (Moylan 2007), only for spherical symmetric lens
   - Born's approximation (Takahashi 2005, Choi 2021) only for weak diffraction $f \to 0$
   - Picard-Lefschetz thimbles (Feldbrugge 2019), choose good **complex contours** only for analytic ψ

2. Time domain methods
   1. Equal time Contour integral (Ulmer 1994, Nakamura 1995, Mishra 2021)
   2. Area between the contours (Diego 2019)

II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

Stationary phase approximation exercise

$$F(w, \mathbf{x}_0) \equiv \frac{w}{2\pi i d_{\rm eff}} \int dx'^2_\perp \exp\left(iwt_d(\mathbf{x}'_\perp)\right)$$

- In $w \to \infty$ limit, only stationary points of $i\, w\, t_d(x'_\perp)$ contributes to the integral

$$\nabla' t_d(\mathbf{x}'_\perp) = \frac{1}{d_{\rm eff}}\left(\mathbf{x}'_\perp - \frac{d_l}{d_s}\mathbf{x}_{0\perp}\right) - \nabla'\psi(\mathbf{x}'_\perp) = 0$$

The solution $\vec{x}_i$ s are the lensing 'image'

- Around the stationary points, the integrand becomes Gaussian

$$\sim e^{iwy^a y^b \partial_a \partial_b t_d}$$

$$\Rightarrow F(w, \mathbf{x}_0) \simeq \sum_i \sqrt{|\mu(\mathbf{x}_i)|} e^{iwt_d(\mathbf{x}_i) - in_i\pi}$$

- $\mu \propto \det(\partial_a \partial_b t_d)^{-1}$: image magnification
- n_i = 0, 1/2 , 1 : Minimum, Saddle, Maximum

II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

Point mass lens example $\psi(\mathbf{x}'_\perp) = GM/c^2 \ln|\mathbf{x}'_\perp|$

- It allows closed expression for the integral (coulomb wave function)

$$F(\omega, y) = e^{\pi\omega/4}\Gamma(1 - i\omega/2)\,{}_1F_1(i\omega/2, 1, i\omega y^2/2)$$

- Stationary phase approximation

$$F(\omega, y) \simeq \sqrt{|\mu_1(y)|} - i\sqrt{|\mu_2(y)|}e^{i\omega\Delta t_{12}(y)}$$

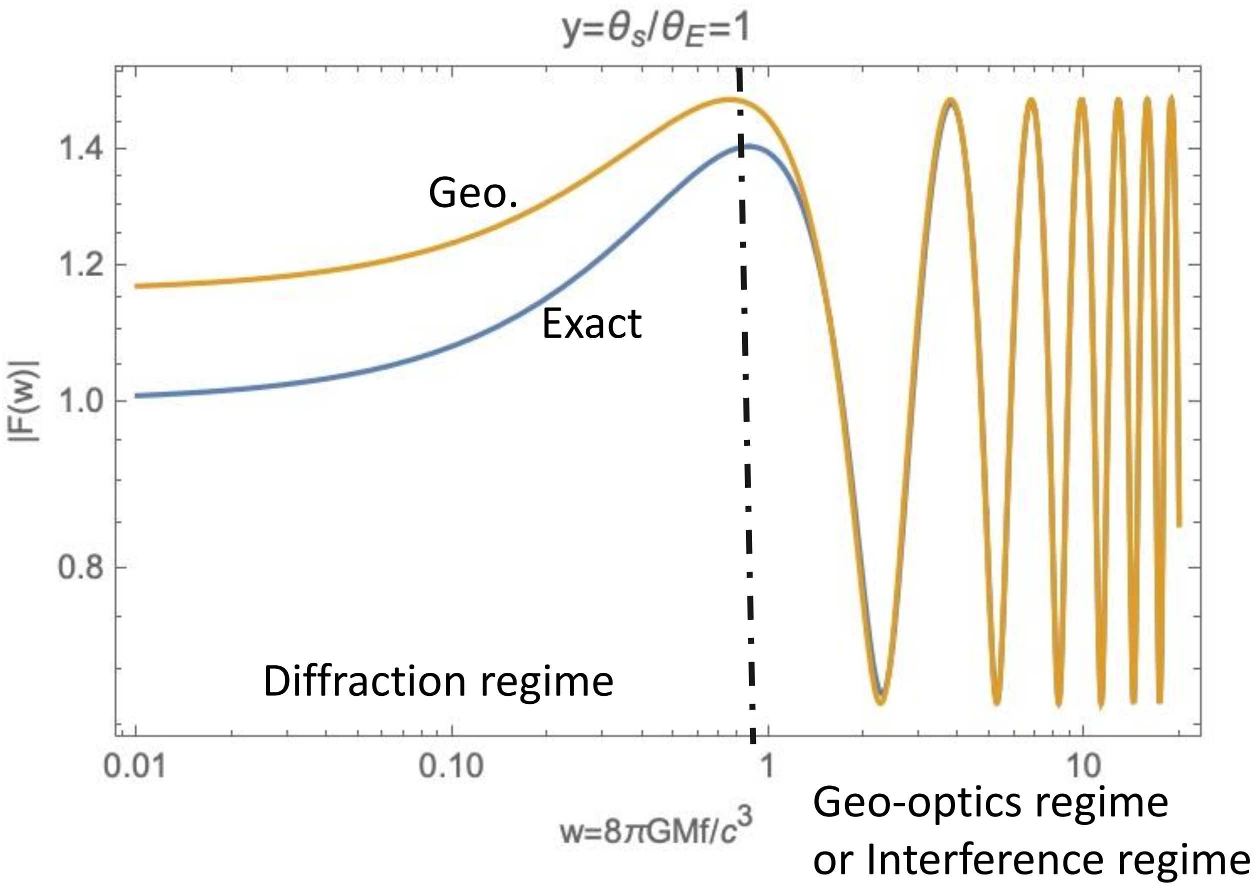
- Normalized variables

$$\omega = 8\pi GMf/c^3$$
$$y = d_l|\mathbf{x}_{0\perp}|/(r_E d_s)$$
$$r_E^2 = 4GM/c^2 \times d_{\text{eff}}$$

II. Wave optics lensing formalism

Solving with Kirchhoff integral theorem

Point mass lens example $\psi(\mathbf{x}'_\perp) = GM/c^2 \ln |\mathbf{x}'_\perp|$



III. Wave optics lensing examples

Detection of Strong lensing event

- Lens : Galaxy or Galaxy cluster ($\tau \sim 10^{-3}$)

- Observables (Hannuksela 2019)
  - Magnification bias – low redshift, higher mass
  - Multiple images - consistent mass, spin … and sky location while different distances and merger time
  - + **Morse phase relations** (Dai 2020)

$$F(\omega, y) \simeq \sqrt{|\mu_1(y)|} - i\sqrt{|\mu_2(y)|}e^{i\omega\Delta t_{12}(y)}$$

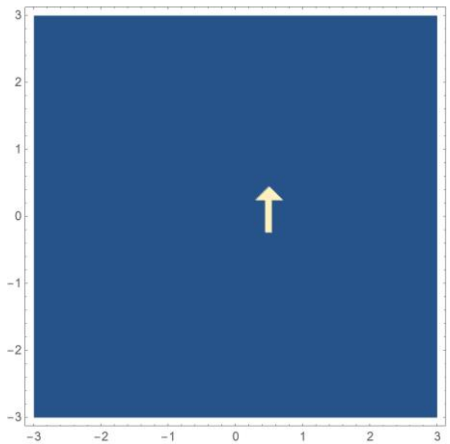
- Using **Morse phase** of GW, we might detect lensing with a **single signal**!

Ezquiaga 2021

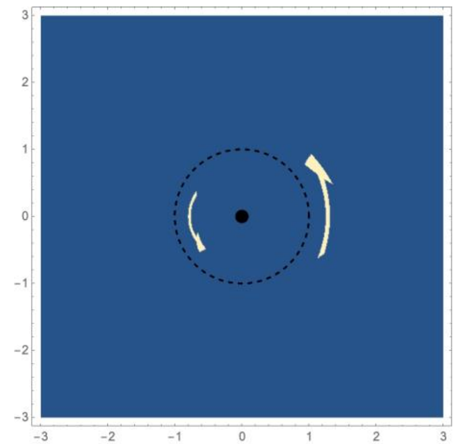
III. Wave optics lensing examples

Detection of Strong lensing event

- In the case of lensing images, It is difficult.



Inverted parity



- In GW, we can use sub-dominant harmonics – unequal mass, eccentric orbit

$$h_{lm} \propto \cos[m(\phi(t) - \phi_0)] \quad \Rightarrow h_{lm} \propto \cos[m(\phi(t) - \phi_0) + n\pi]$$

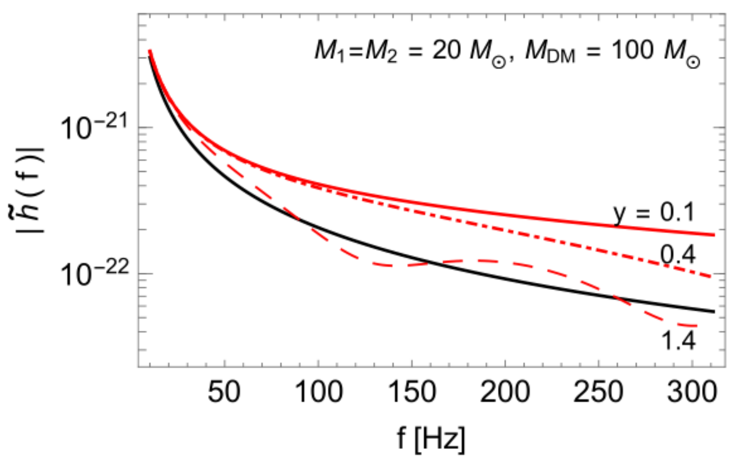
- Orbital phase shift cannot mimic the Morse phase

III. Wave optics lensing examples

Detection of Microlensing interference

- Lens : Intermediate mass black holes $10^2\ M_{\odot}{\sim}10^4 M_{\odot}$ Jung 2019
- Optical depth highly depends on IMBH population (cf) Primordial black holes)
- Observables
  - Modulation in amplitude and phase

$$h_L(f) = F(f)h(f)$$



Jung 2019

III. Wave optics lensing examples

Point mass lens example

$$r_{\rm sch} = 3 \times 10^5 \ \mathrm{km}$$

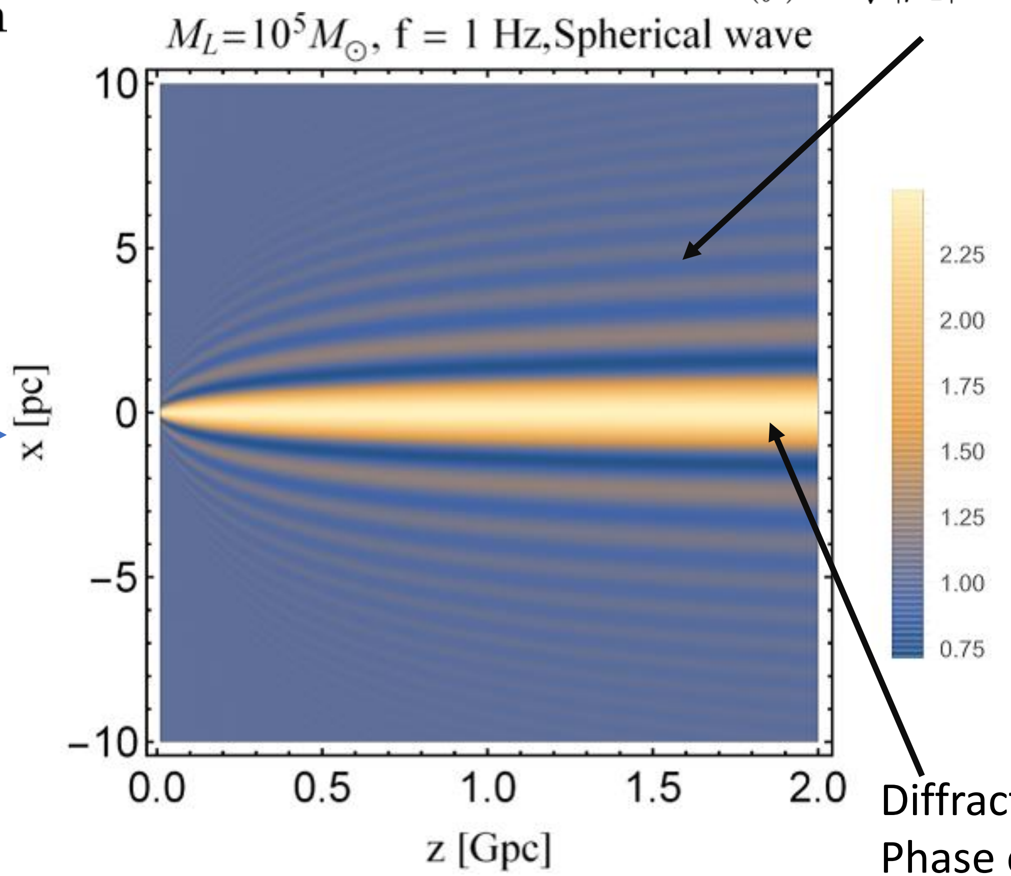
$$\lambda = 3 \times 10^5 \ \mathrm{km}$$

Source

1 Gpc from the lens

Geometric optics

$$F(f) \simeq \sqrt{|\mu_1|} - i\sqrt{|\mu_2|}e^{i2\pi f \Delta T_{12}}$$



$|F(f)|$

Diffraction,
Phase coherence region

III. Wave optics lensing examples

Point mass lens example

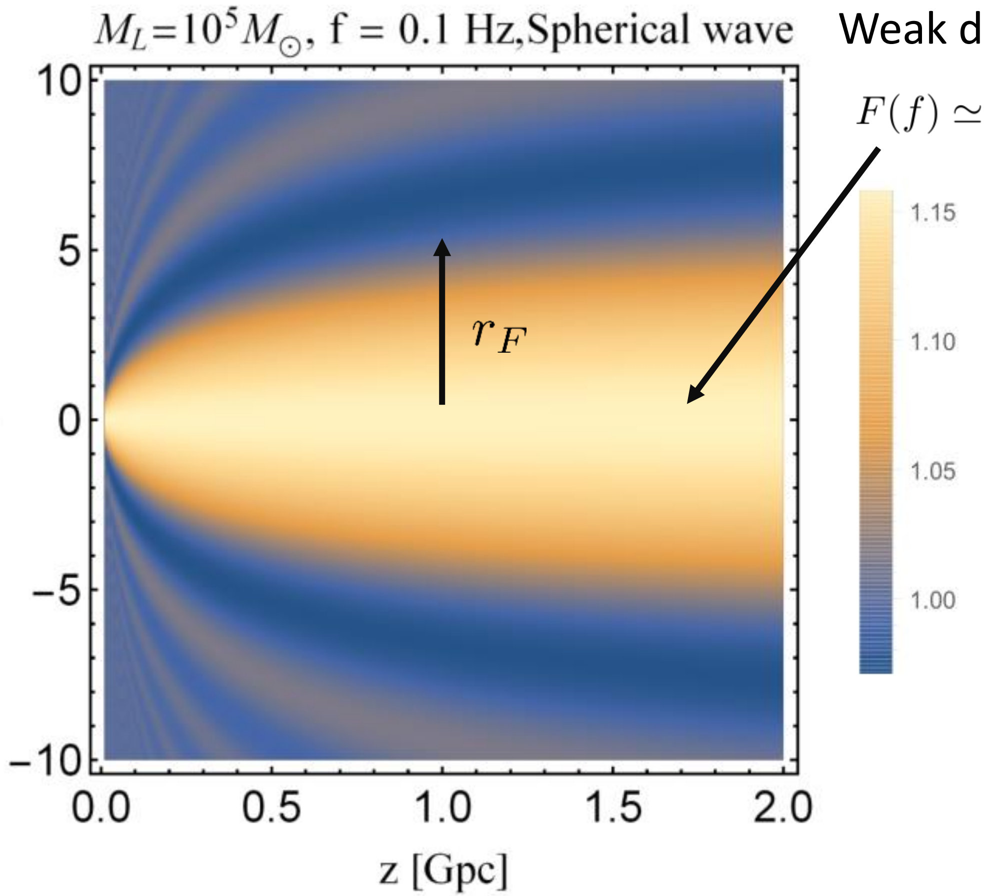
Fresnel length $r_F = \sqrt{\frac{d_{\rm eff}}{\pi f}} \simeq 5.56{\rm pc}\sqrt{\left(\frac{d_{\rm eff}}{\rm Gpc}\right)\left(\frac{0.1{\rm Hz}}{f}\right)}$

Weak diffractive lensing

$$F(f) \simeq 1 + \bar{\kappa}\left(\frac{r_F}{\sqrt{2}} e^{i\pi/4}\right)$$

Source

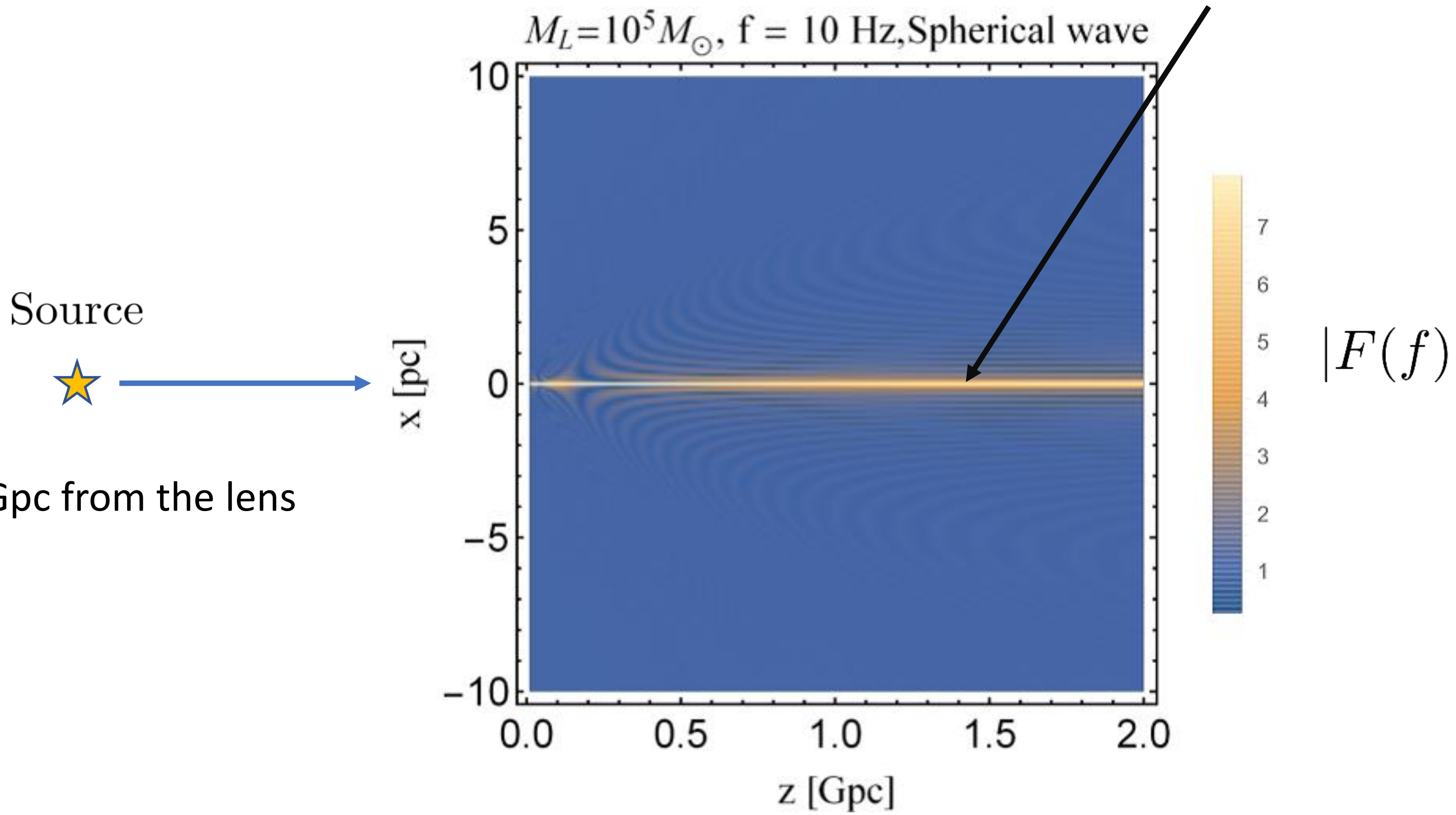
1 Gpc from the lens



III. Wave optics lensing examples

Point mass lens example

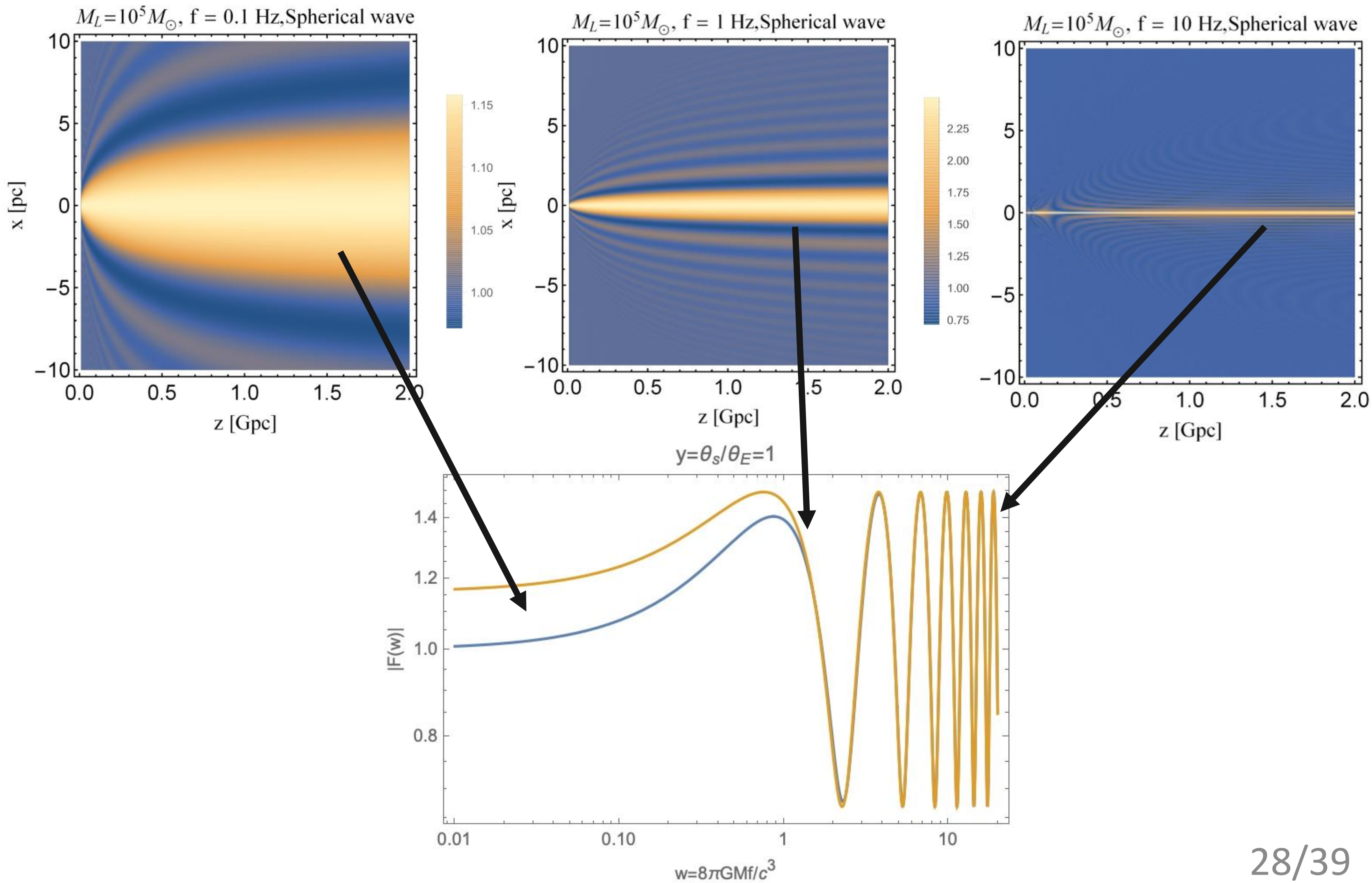
When $r_E > r_F$, it is Strong diffraction $r_S = \frac{r_F}{r_E} r_F < r_F$



1 Gpc from the lens

Einstein radius $r_E = \sqrt{4\frac{GM}{c^2} d_{\rm eff}} \simeq 4.4\ {\rm pc} \sqrt{\left(\frac{M}{10^5 M_\odot}\right)\left(\frac{d_{\rm eff}}{1\ {\rm Gpc}}\right)}$

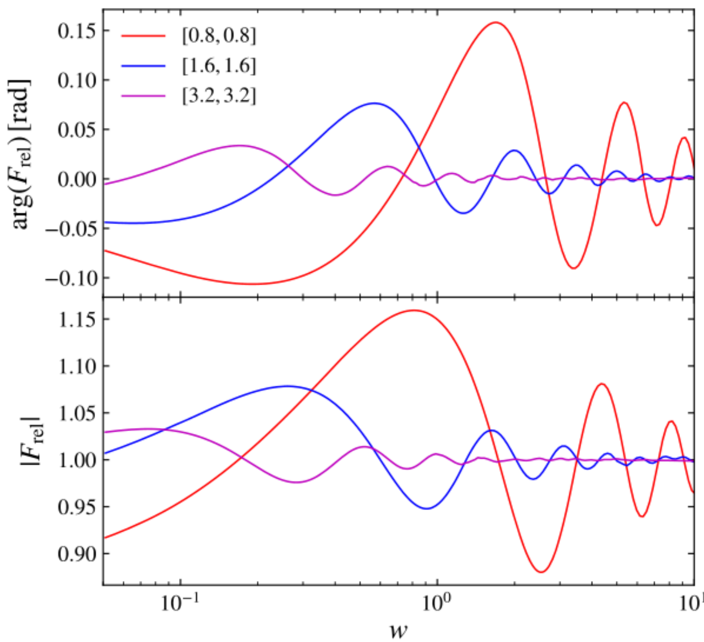
III. Wave optics lensing examples



III. Wave optics lensing examples

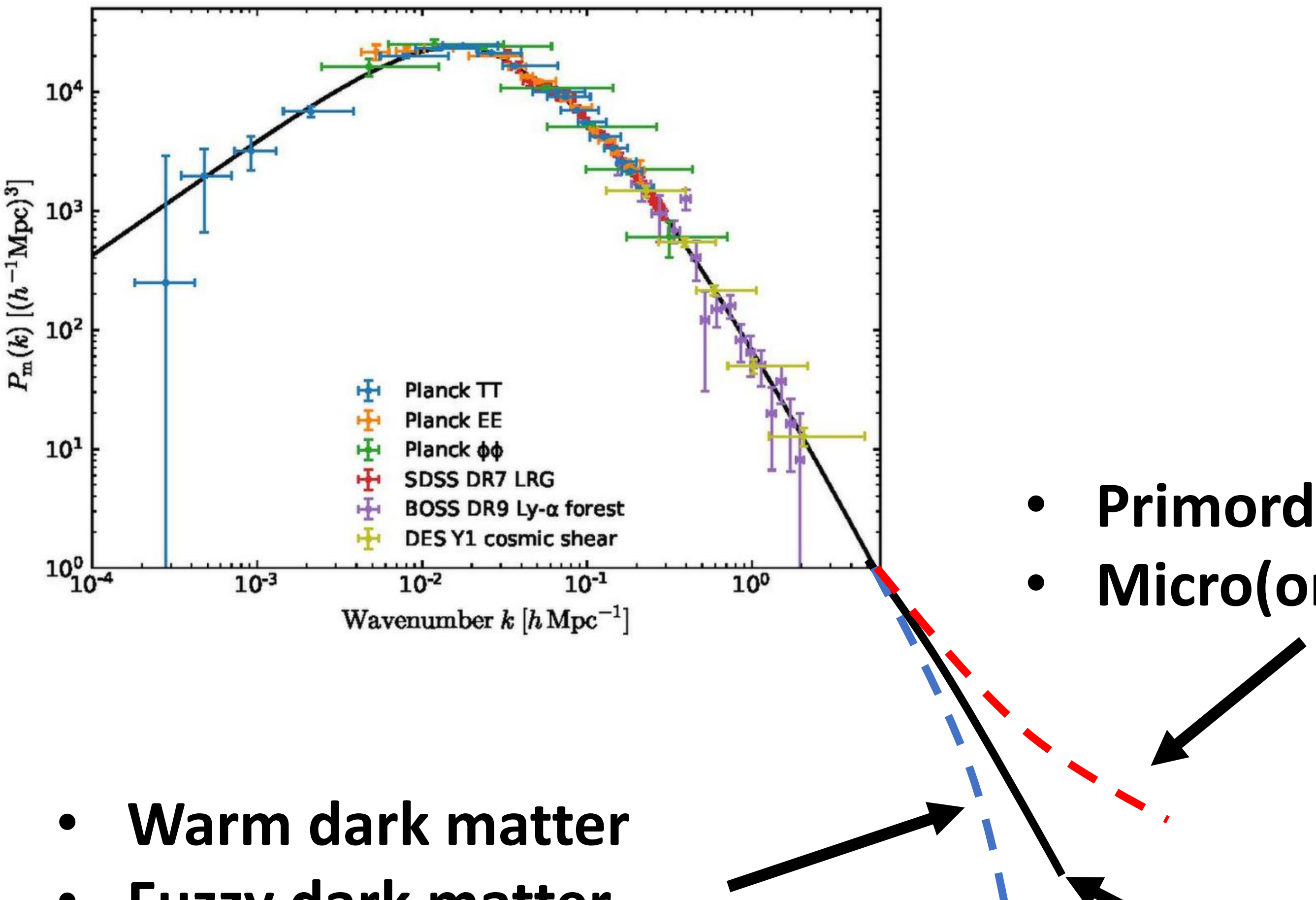
Detection of Diffraction only lensing

- Lens : Singular Isothermal Sphere (~dwarf galaxy) $10^2\ M_\odot \sim 10^3 M_\odot$ Dai 2018
- No multiple image = No interference
  - Almost Impossible to detect with Electromagnetic signals
- Observables
  - Modulation in amplitude and phase



IV. Probing dark matter halo

Cosmological structures at small scale depends on Dark matter physics.



- **Primordial Black holes**
- **Micro(or Mini) halos**

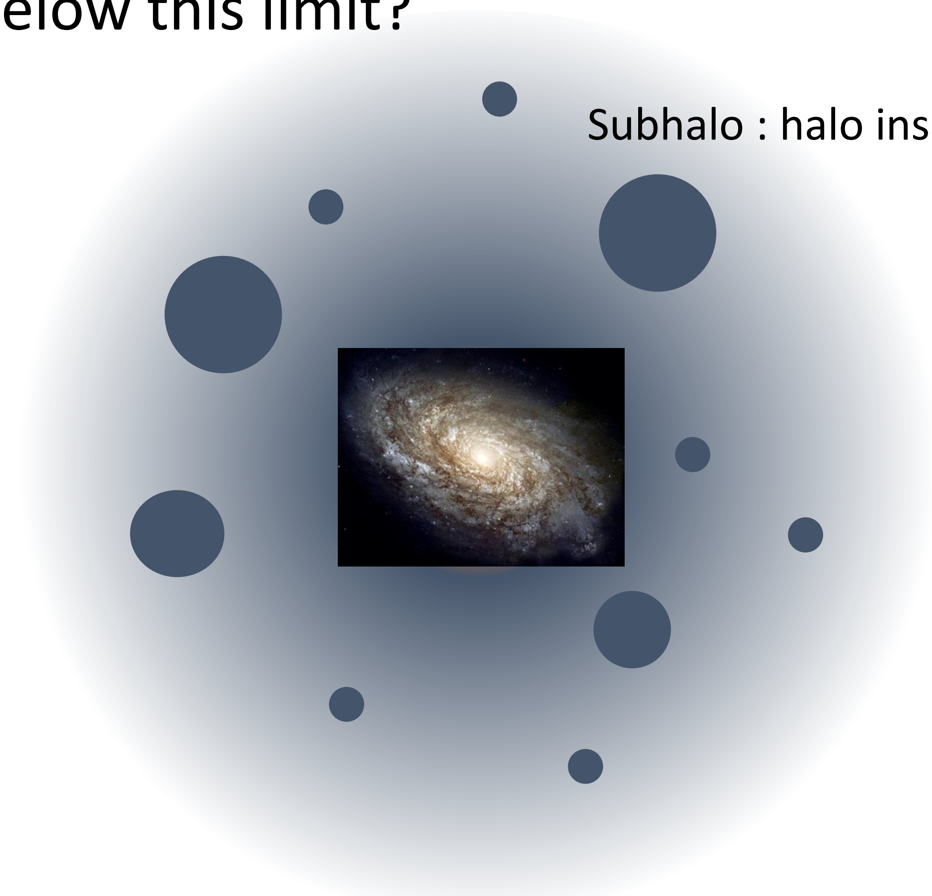
- **Warm dark matter**
- **Fuzzy dark matter**
- **Self-Interacting dark matter**

CDM

IV. Probing dark matter halo

Probing dark matter subhalo is the best option for small scale until now.

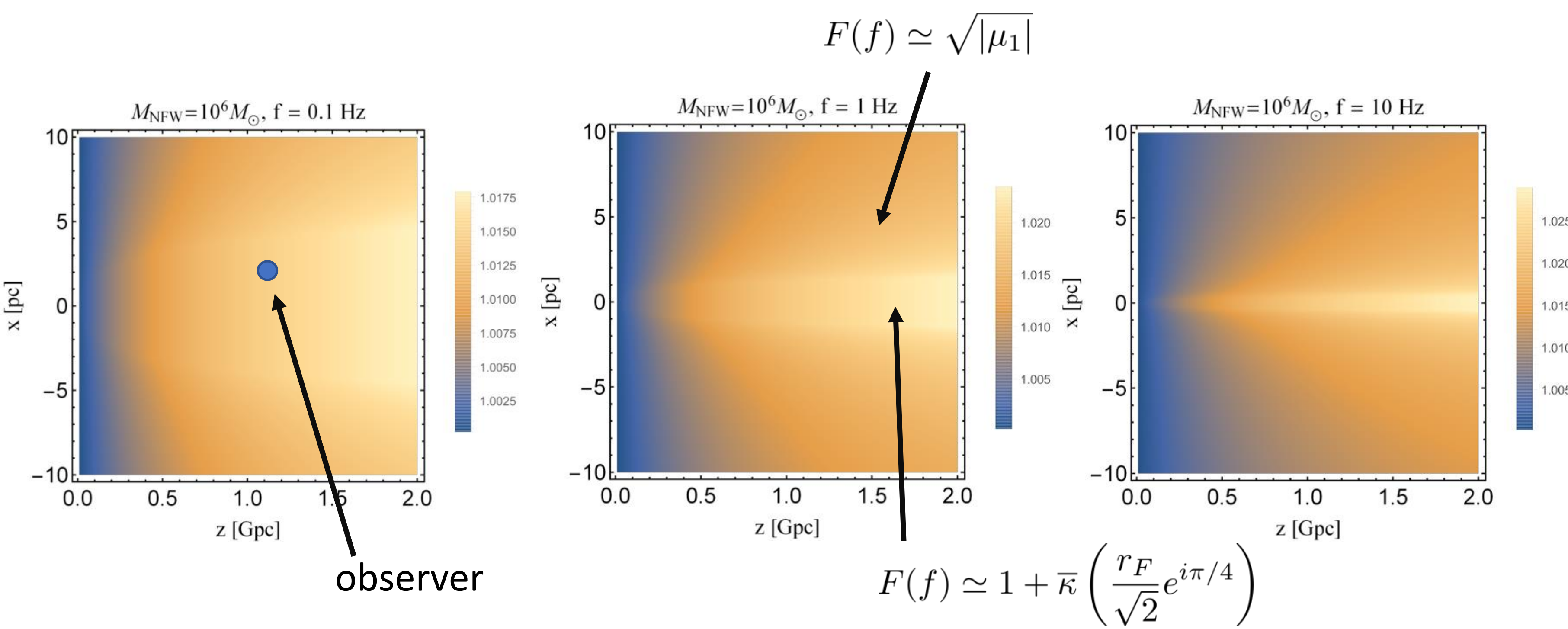
- Current limit : $\mathrm{M_{sub}} \sim 10^7 M_{\odot}$ $(k = 10^{3\sim4} Mpc^{-1})$ (Nadler 2021)
- Can we go below this limit?



IV. Probing dark matter halo

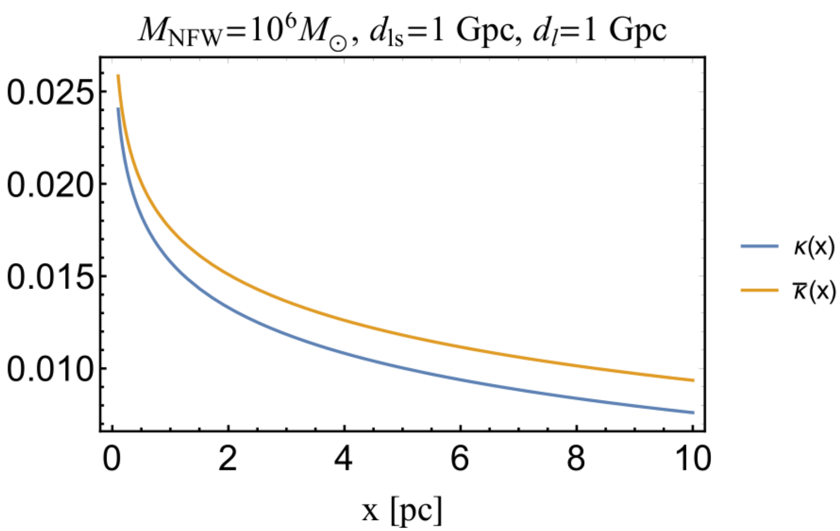
Lensing by dark matter halo (Navarro-Frenk-White)

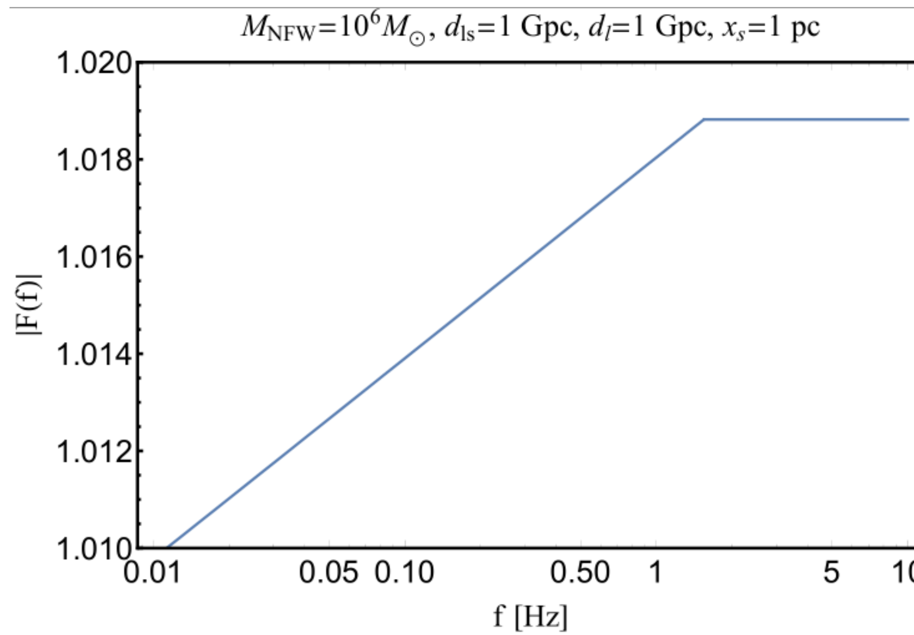
- Self gravitating, but very diffuse mass distribution
- Very weak gravitational potential, no multiple image
- It has zero Einstein radius i.e. always weak diffraction



IV. Probing dark matter halo

We can use GW chirps to probe diffractive lensing!

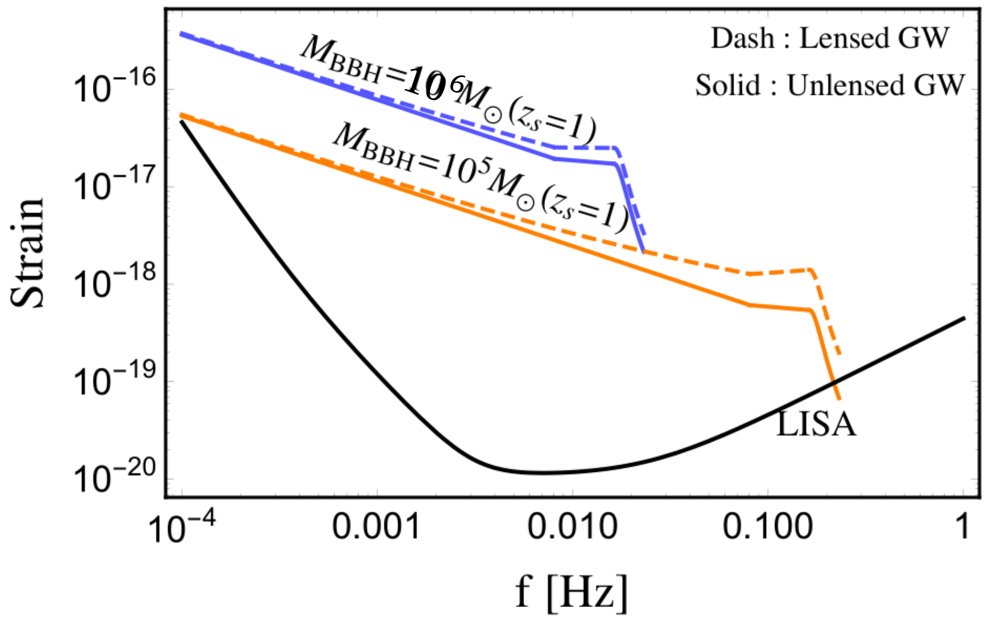




$$F(f) \simeq 1 + \overline{\kappa}\left(\frac{r_F(f)}{\sqrt{2}} e^{i\pi/4}\right)$$

IV. Probing dark matter halo

Weak Diffractive lensing of gravitational wave



Although we don't know intrinsic luminosity of GW, this Frequency dependent amplification can be detected.

Lensing by $\bar{\kappa}(r) \propto r^{-1}$ lens ($M = 10^5\ M_\odot, z_l = 0.35$)

We can measure the difference by log-likelihood of GWs

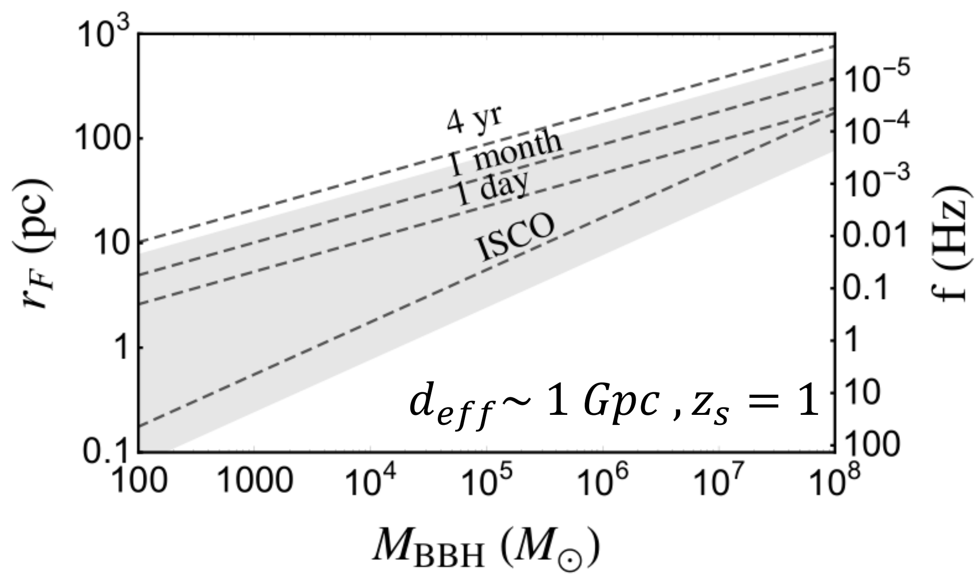
$$\ln p = -\frac{1}{2} \min_{t_c,\phi_c} (h_L - h_0 | h_L - h_0) \qquad (h_1|h_2) = 4\mathrm{Re} \int df \frac{h_1^*(f) h_2(f)}{S_n(f)}$$

IV. Probing dark matter halo

Weak Diffractive lensing of gravitational wave

GW chirps from **massive Black hole binaries** is ideal diffractive lensing source
: low f, large d_{eff} → large r_F

$$r_F \simeq 5.56\text{pc}\sqrt{\left(\frac{d_{\text{eff}}}{\text{Gpc}}\right)\left(\frac{0.1\text{Hz}}{f}\right)} \sim (\text{sub halo length scale})$$



Ex) $10^5 M_\odot$ BBH spectrum → Scan 1 pc to 50 pc by 1yr observation

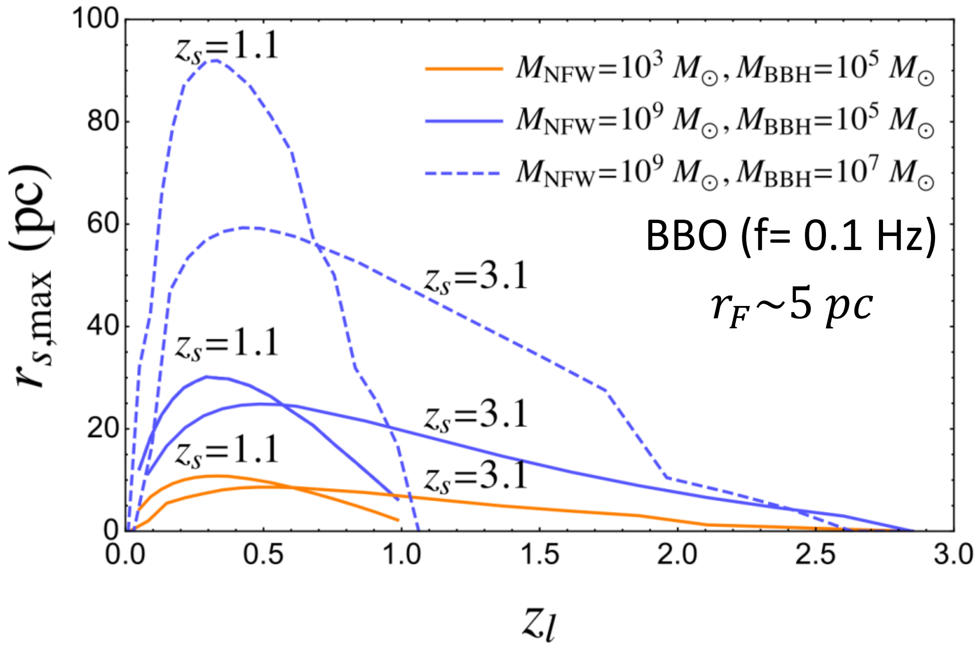
IV. Probing dark matter halo

Set 3-sigma criteria to lnp, we find maximum impact parameter 'y'-> cross-section

Lensing cross-section : shear at r_F

⇒Insensitive to mass at high SNR limit

$$|\ln p| \simeq \frac{1}{8}\left\{\rho_0 \cdot \left|\gamma\left(\frac{r_F(f_0)e^{i\frac{\pi}{4}}}{\sqrt{2}}\right)\right| \cdot \ln\frac{f_{\max}}{f_{\min}}\right\}^2$$



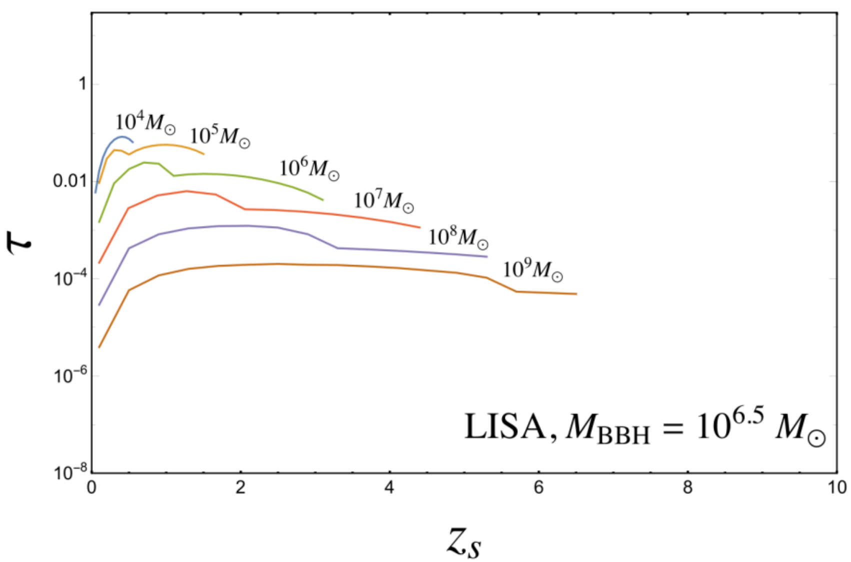
Mass scale difference : 10^6

Cross-section scale difference : $O(1)$

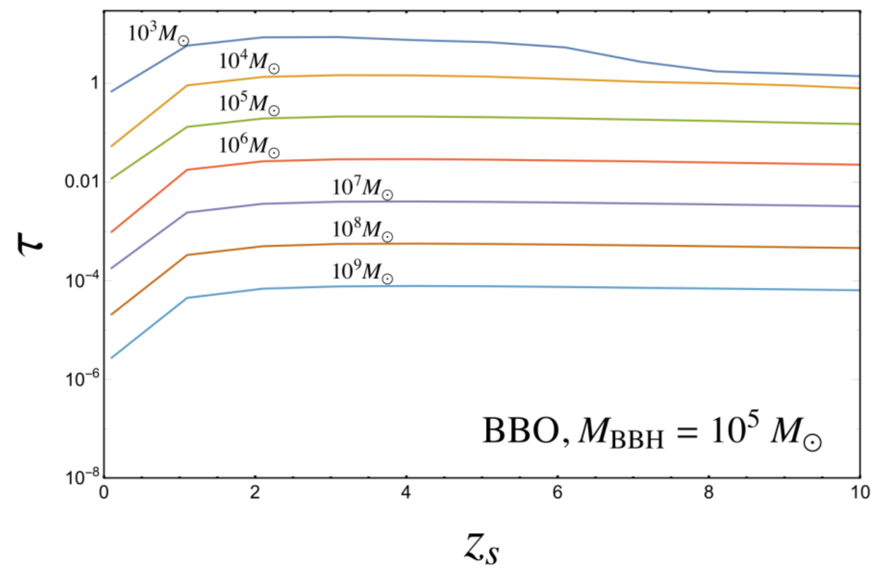
IV. Probing dark matter halo

Optical depth from the lensing cross-section

$\tau \sim 0.01$ for $10^6 M_\odot$ halo



$\tau > 1$ for $10^{3-4} M_\odot$ halo



$$\tau \propto (\text{number of Lens}) \propto 1/(M_{\rm halo})$$

Diffractive lensing is sensitive to low mass halo !

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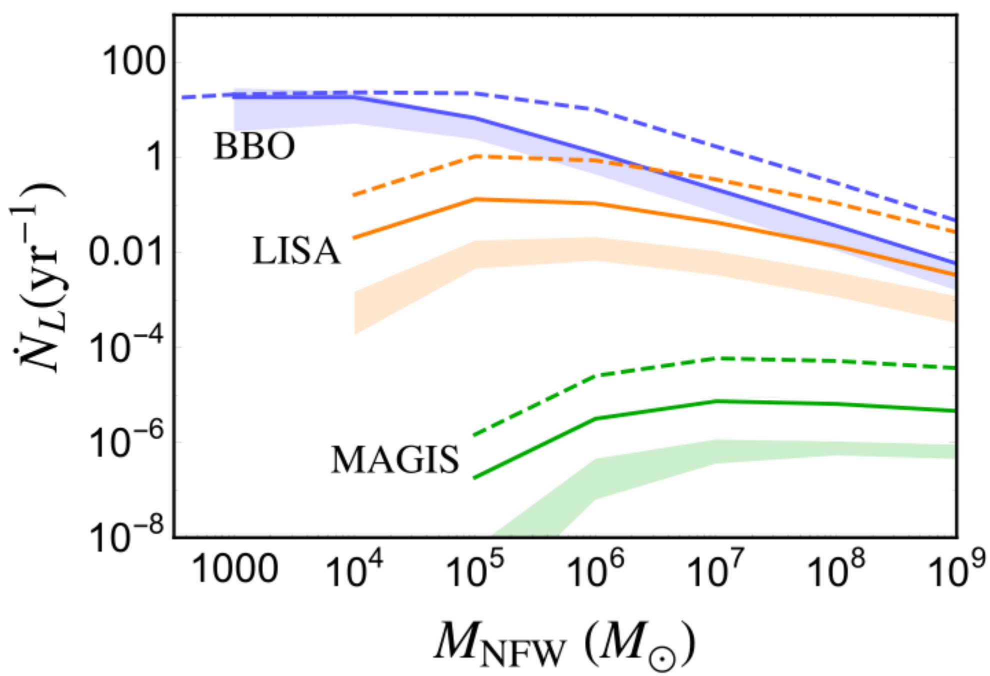
Prospect

We need powerful space-based GW detectors like BBO(0.1Hz), LISA(1 mHz).

- BBO can detect $10^{3-4}M_{\odot}$ halo lensing O(10) events per year.

  In future, BBO will discriminate CDM and the other DM models.

- LISA and the others are less promising.
  - Lack of **High Signal-to-Noise Ratio(>1000)** BBH sources



BBH Merger rate
Solid : 0.01 $Gpc^{-3}yr^{-1}$
Shaded : astrophysical
(Bonetti 2018)

Summary

1. GW has many good properties to detect wave optics effects of lensing.

2. Wave optics of lensing can be described by Kirchhoff integral, and there are many efforts to solve the integral.

3. Lensing of GWs can provide unique observables through wave optics effects (Amp. and phase modulation)

4. The future powerful GW detector can detect $10^{3\sim4}M_{\odot}$ DM halo through diffractive lensing, which is almost impossible for EM observations.

Thank you !