# Stochastic Gravitational Waves and Its Detection

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The 69th Workshop on Gravitational Waves and Numerical Relativity

# Motivations

- The discovery of stochastic gravitational waves background (SGWB) from supermassive binaries is just around the corner.
- The second wave of GW astronomy is coming.
- I expect that SGWB will be the basic course of GW science.
- I will give an intensive basic lecture of SGWB for astrophysicists including the properties of stochastic GWs and working principles of its detectors.

# Overview

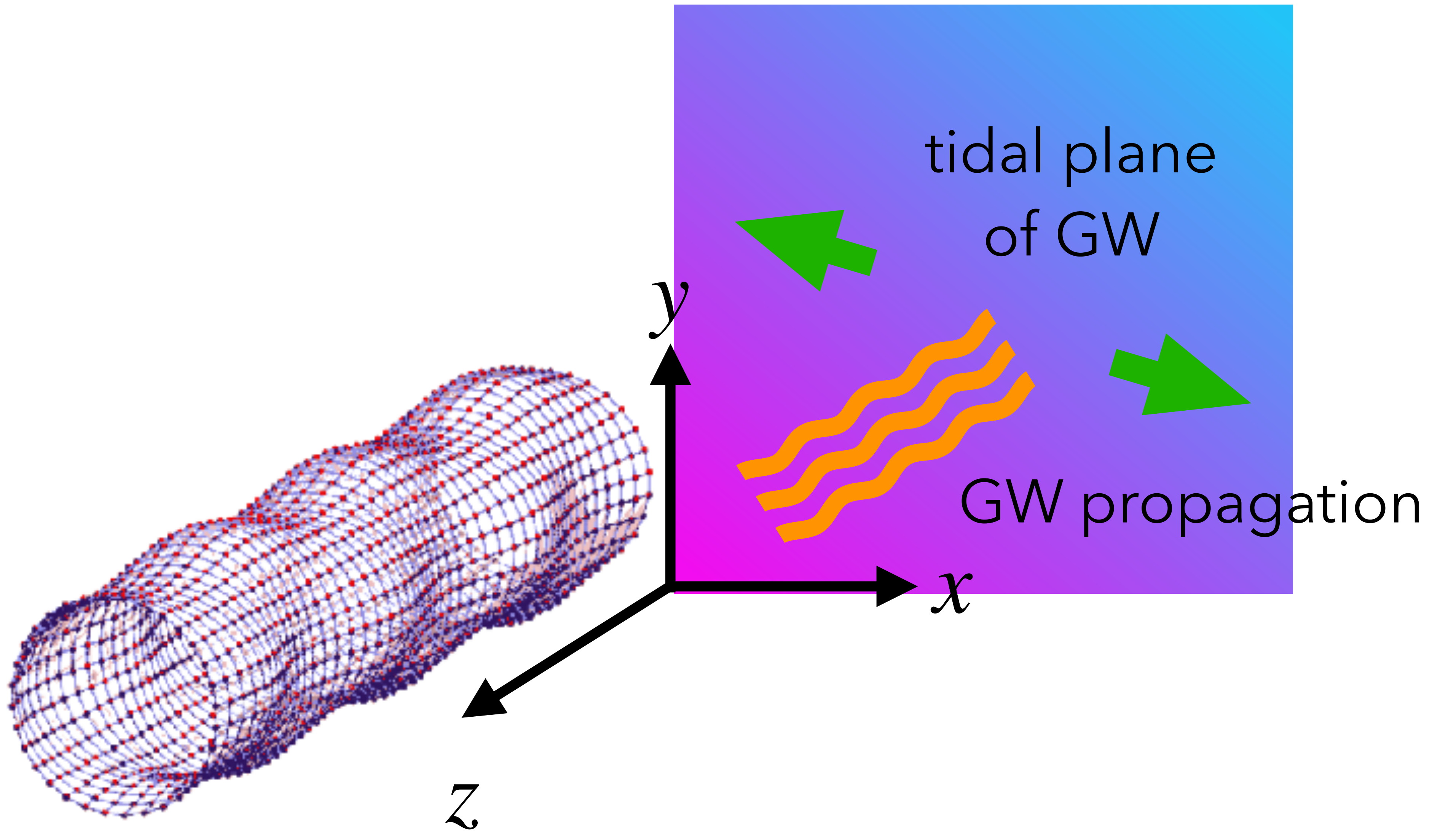
- Stochastic Gravitational Waves (SGWs)
- Noise Reduction by Correlation Method
- Pulsar Timing Arrays (PTAs)
- Detection of SGWs by Electromagnetic Cavities

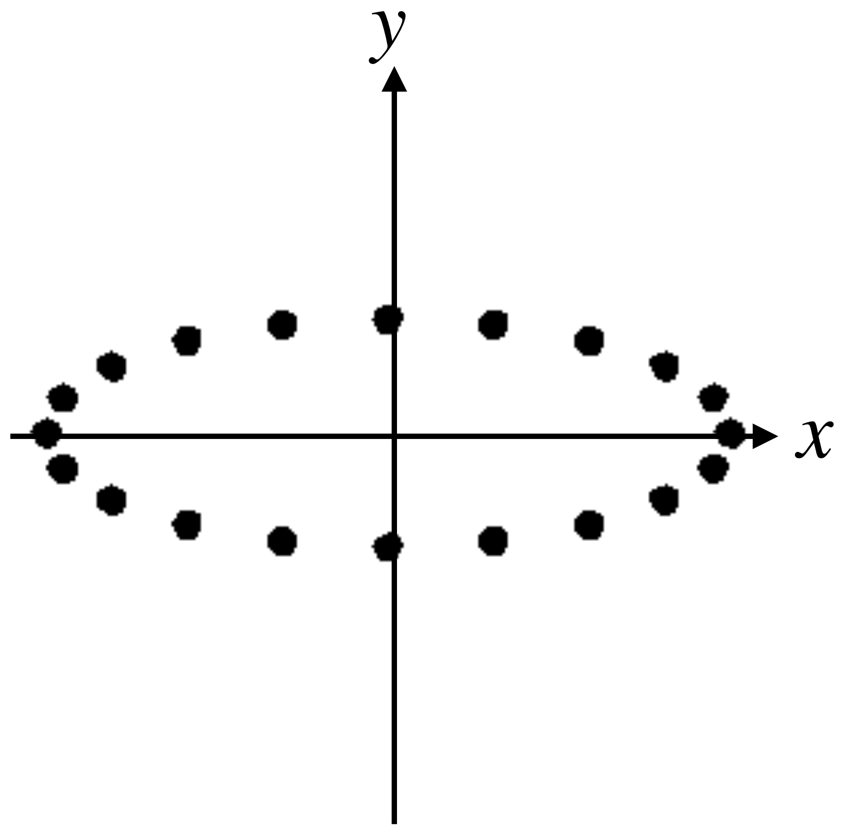
uGMRT / Pune, India

# Gravitational Waves

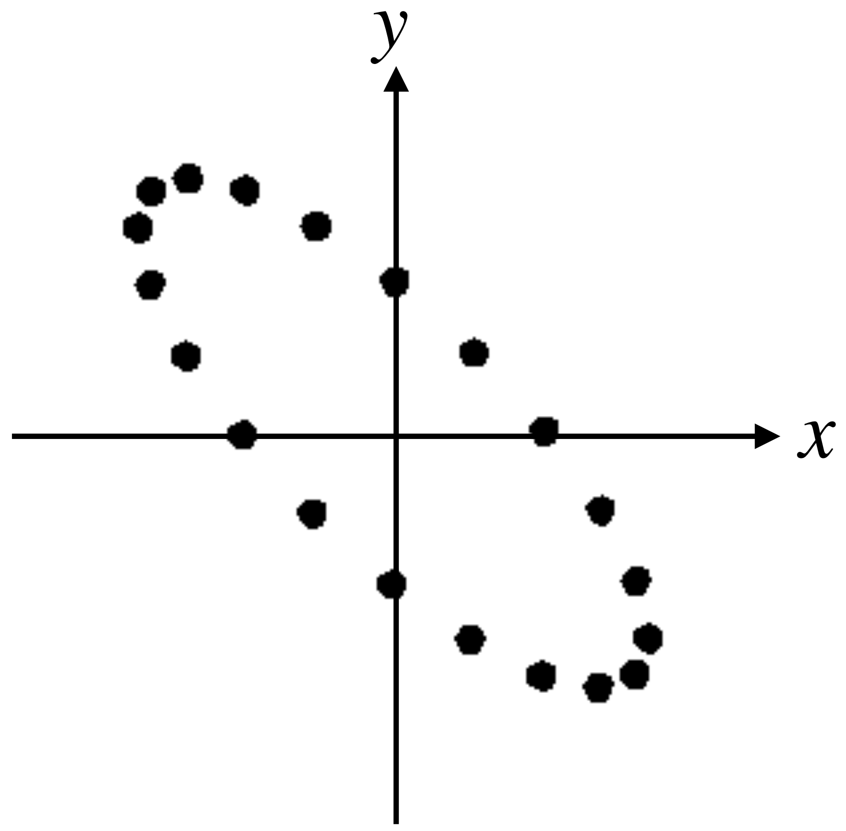
# Properties of Gravitational Waves (GWs)

- Propagation speed: speed of light
- Transverse-Traceless Gauge
  - Transverse wave: propagation direction ⊥ tidal direction
  - No expansion of tidal plane: GWs do not change the area, but the shape.
  - Two polarization modes: plus polarization and cross polarization



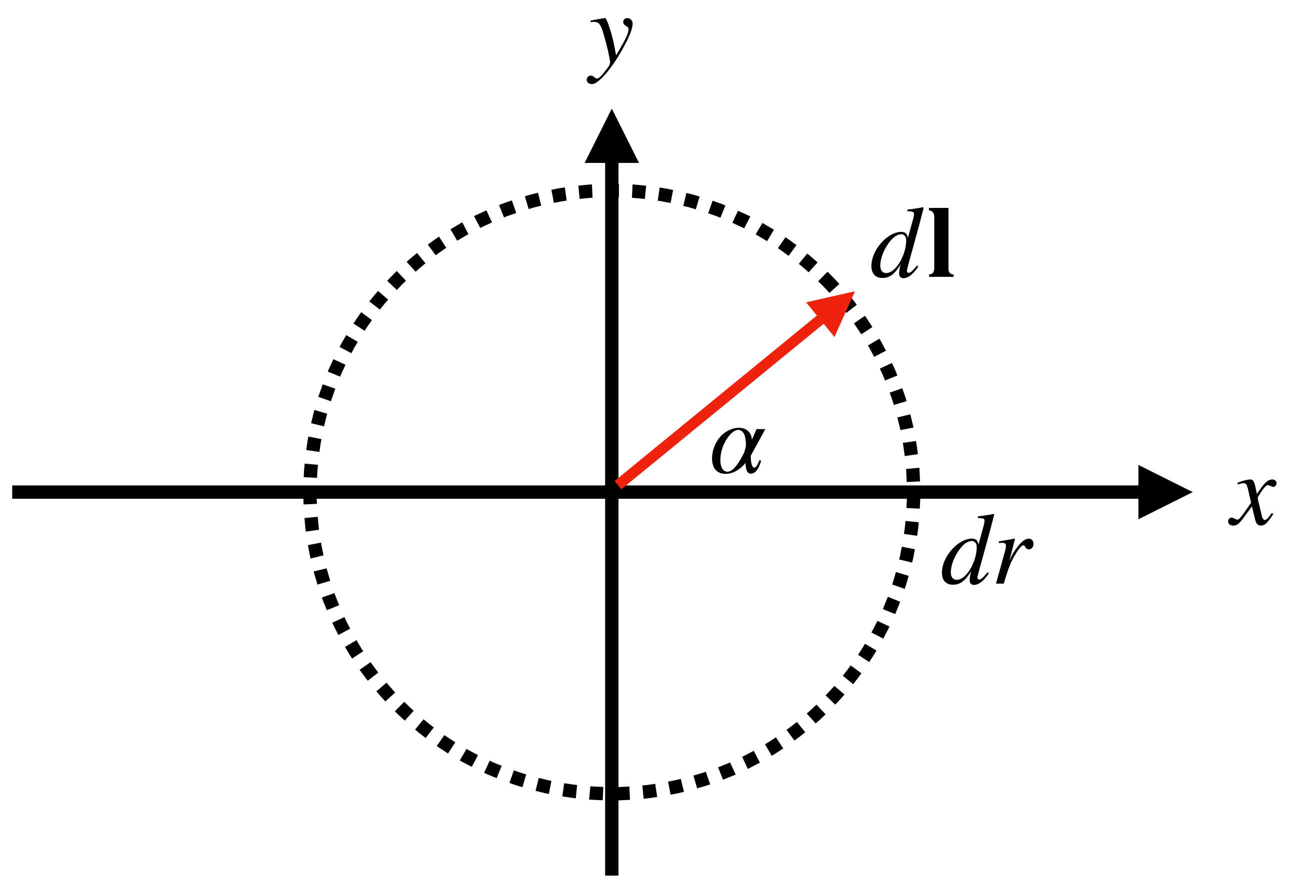

+ polarization




× polarization

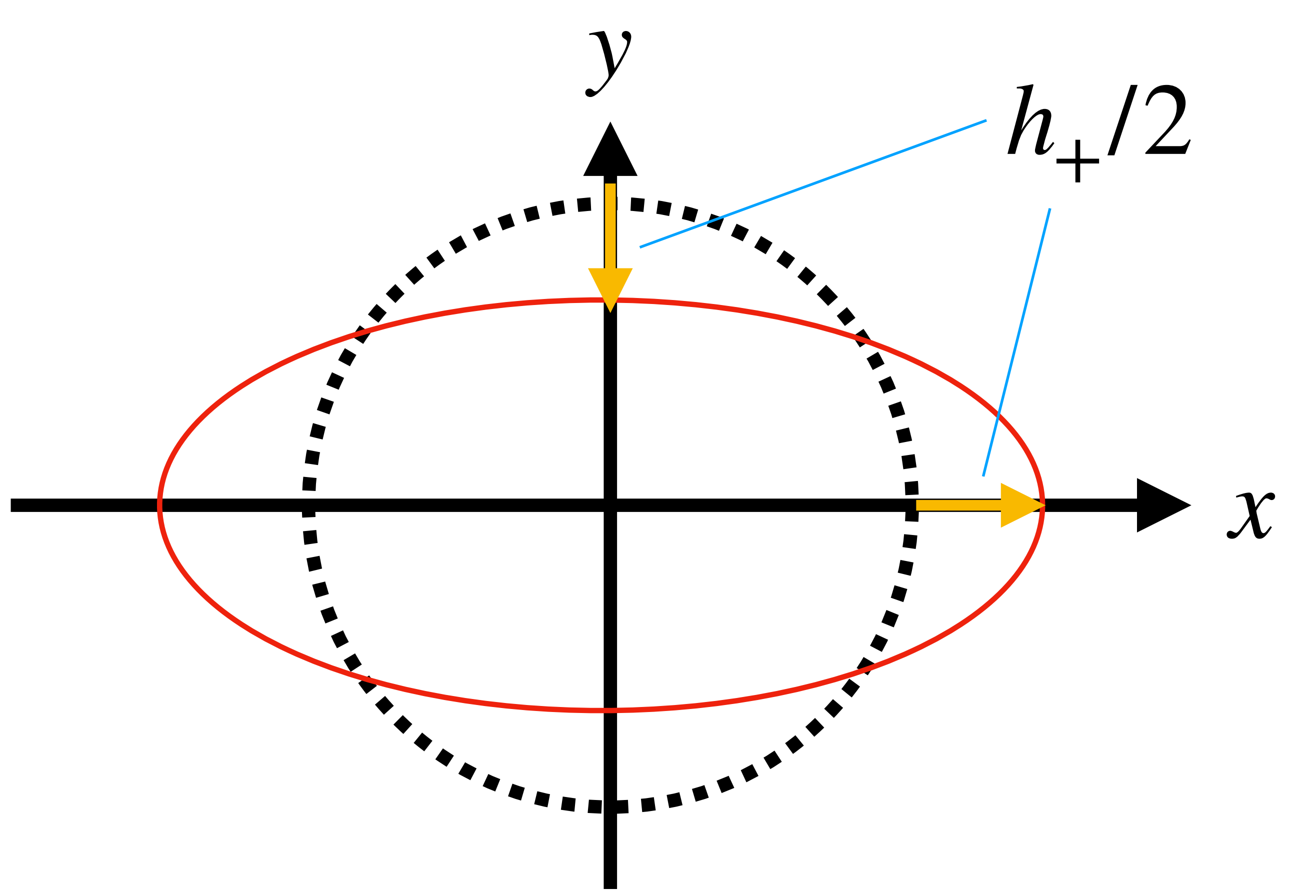
# Metric Perturbation of GWs

- Let us consider a metric perturbation $h_{\mu\nu} \ll 1$ preserving area of tidal plane
- $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- Let us consider the coordinate displacement $d\mathbf{l} = \left(0, dr\cos\alpha, dr\sin\alpha, 0\right)$

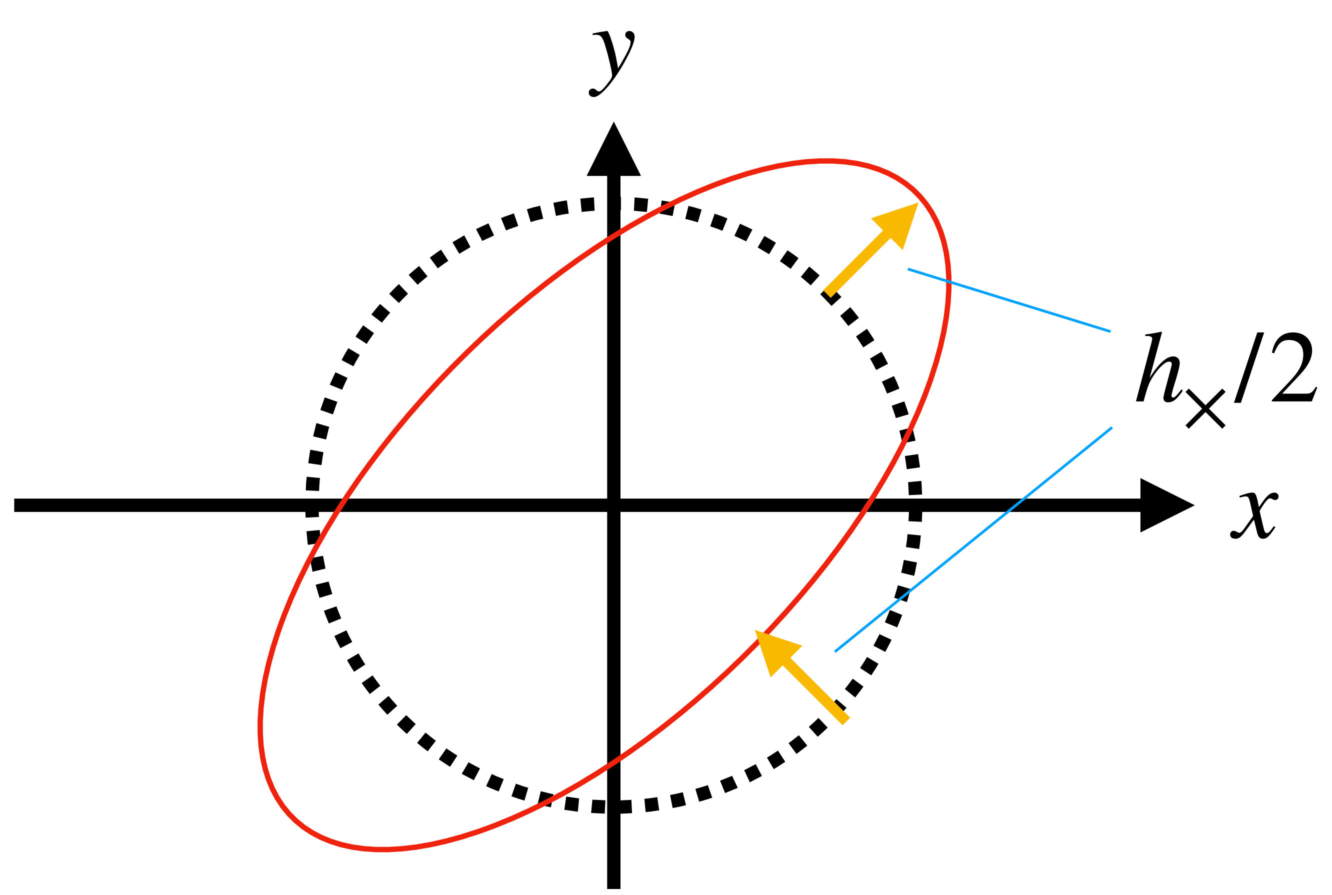


# Metric Perturbation of GWs

- The physical length of $d\mathbf{l}$ is given by
- $$d\mathbf{l}\cdot d\mathbf{l} = [0 \quad dr\cos\alpha \quad dr\sin\alpha \quad 0]\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1+h_+ & h_\times & 0\\ 0 & h_\times & 1-h_+ & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ dr\cos\alpha\\ dr\sin\alpha\\ 0\end{bmatrix}$$
- $$\sqrt{d\mathbf{l}\cdot d\mathbf{l}} = dr\left\{1+\frac{1}{2}h_+\cos(2\alpha)+\frac{1}{2}h_\times\sin(2\alpha)\right\}$$



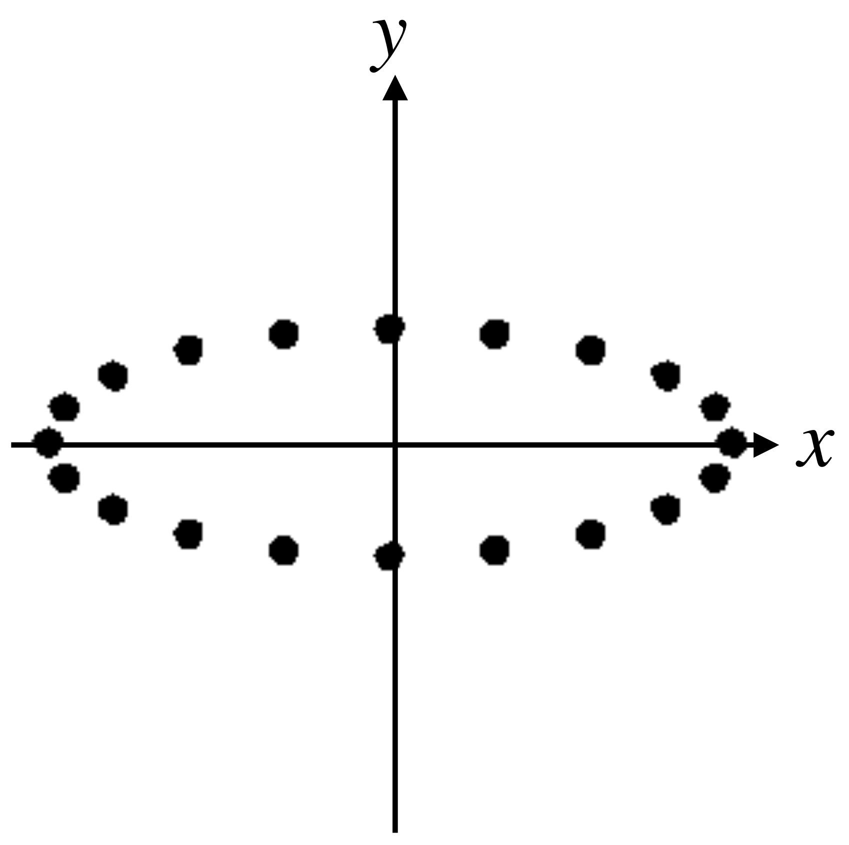
+ polarization



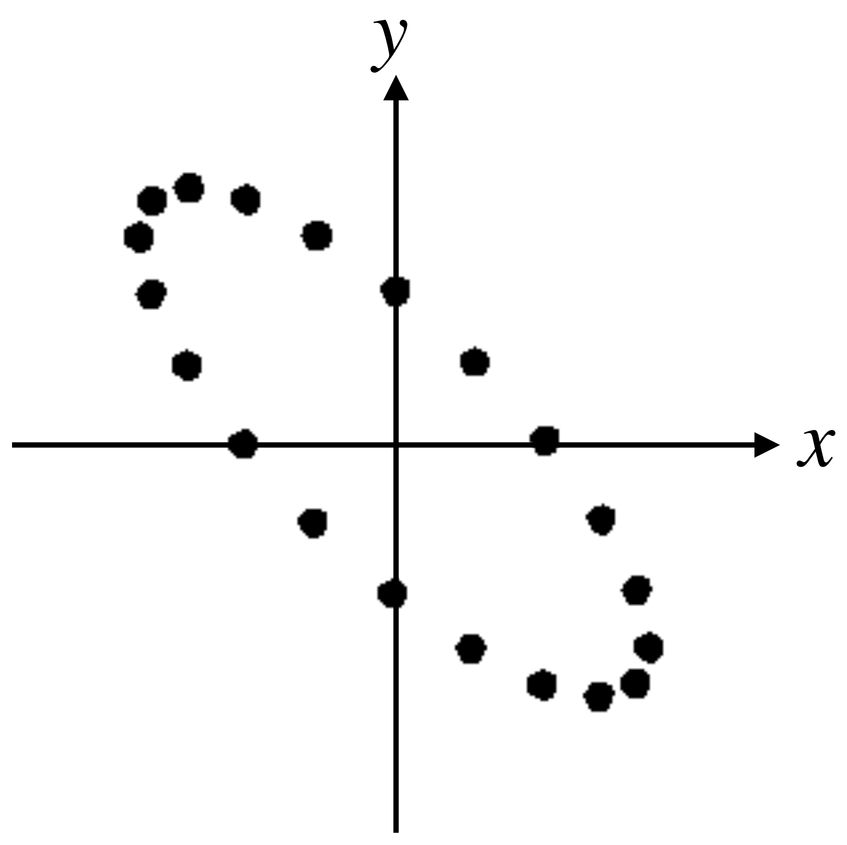
x polarization

# Gravitational Waves

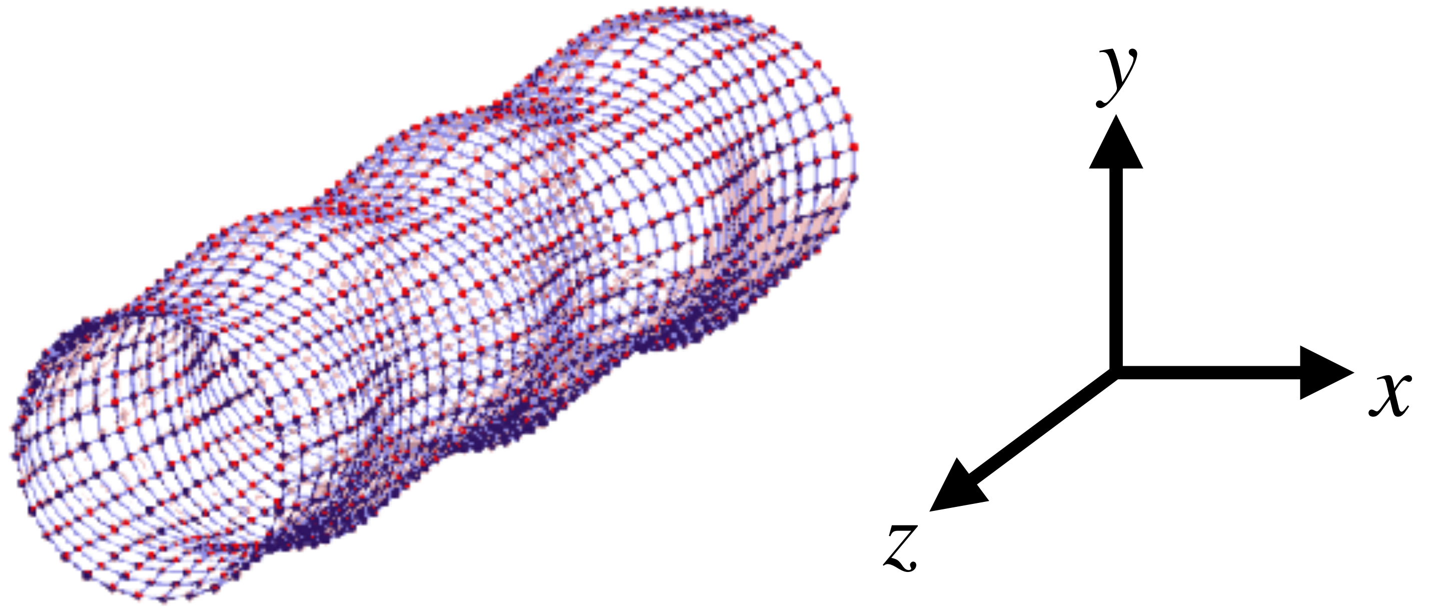
- Monochromatic Plane GWs propagating to $+z$ axis
- $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos\left\{\omega_{\mathrm{g}}(-t+z)+\phi\right\}$$




+ polarization




× polarization

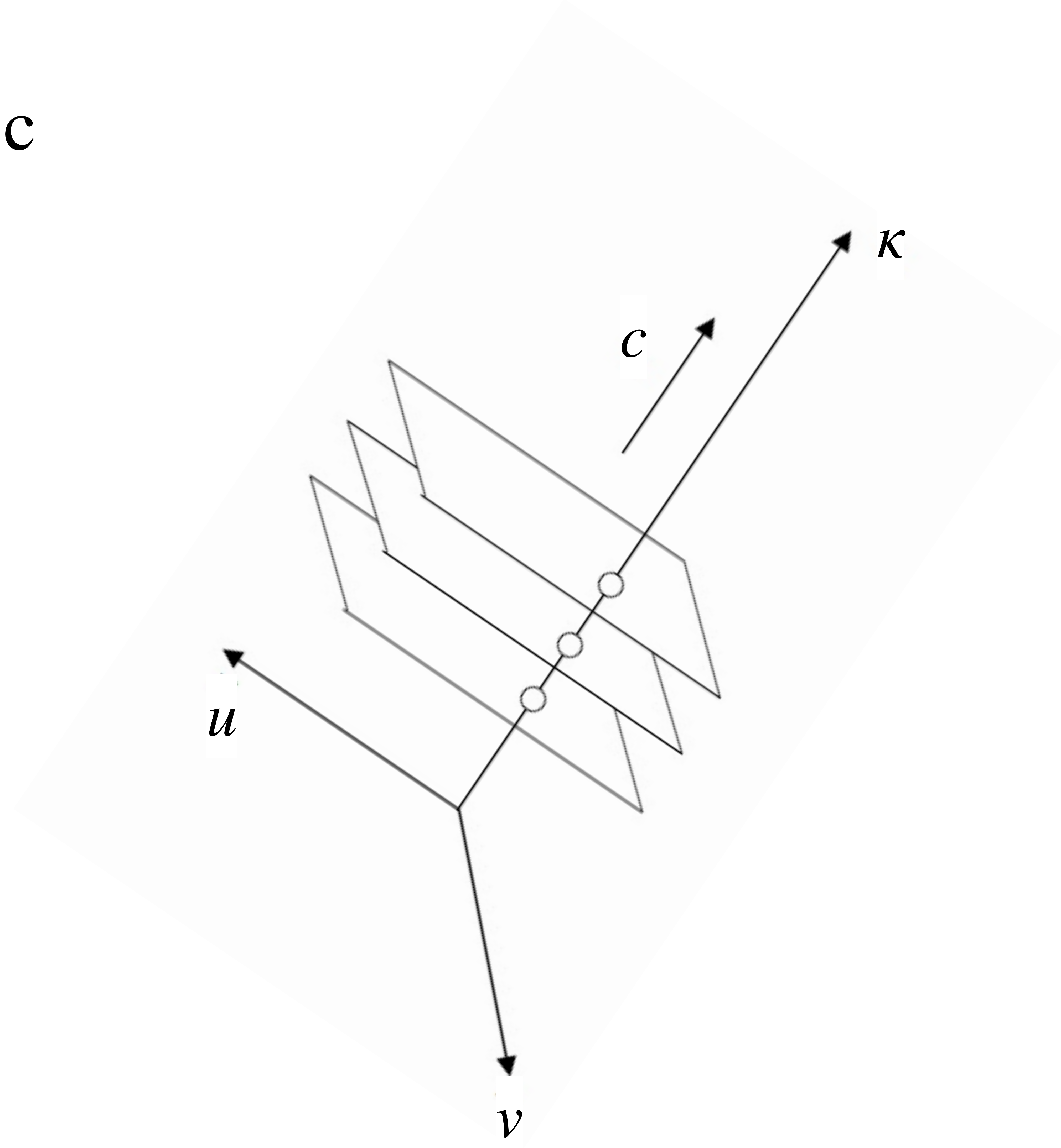



propagation of GWs

# Stochastic Gravitational Waves

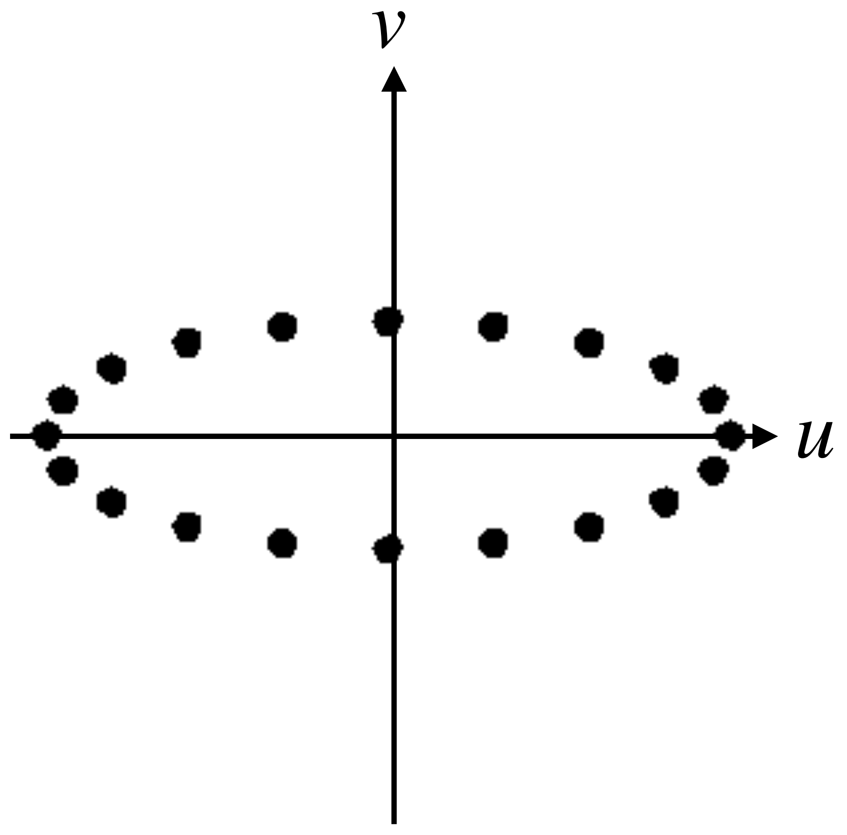
# Monochromatic Plane GWs

- $h_{ab}\left(t,\vec{x}\right) = 2\Re\left[\tilde{h}_{ab}e^{iP\left(t,\vec{x}\right)}\right] = \tilde{h}_{ab}e^{iP\left(t,\vec{x}\right)} + \mathrm{c.c}$
- Complex amplitude: $\tilde{h}_{ab}$
- Phase: $P\left(t,\vec{x}\right) = \omega\left(-t + \kappa\cdot\vec{x}\right)$
- Angular Frequency: $\omega$
- Unit vector of propagation direction: $\kappa$
- Transverse-Traceless Gauge condition
  - $\tilde{h}^a{}_a = 0 \qquad \tilde{h}_{ab}n^b = 0 \qquad \tilde{h}_{ab}\kappa^b = 0$

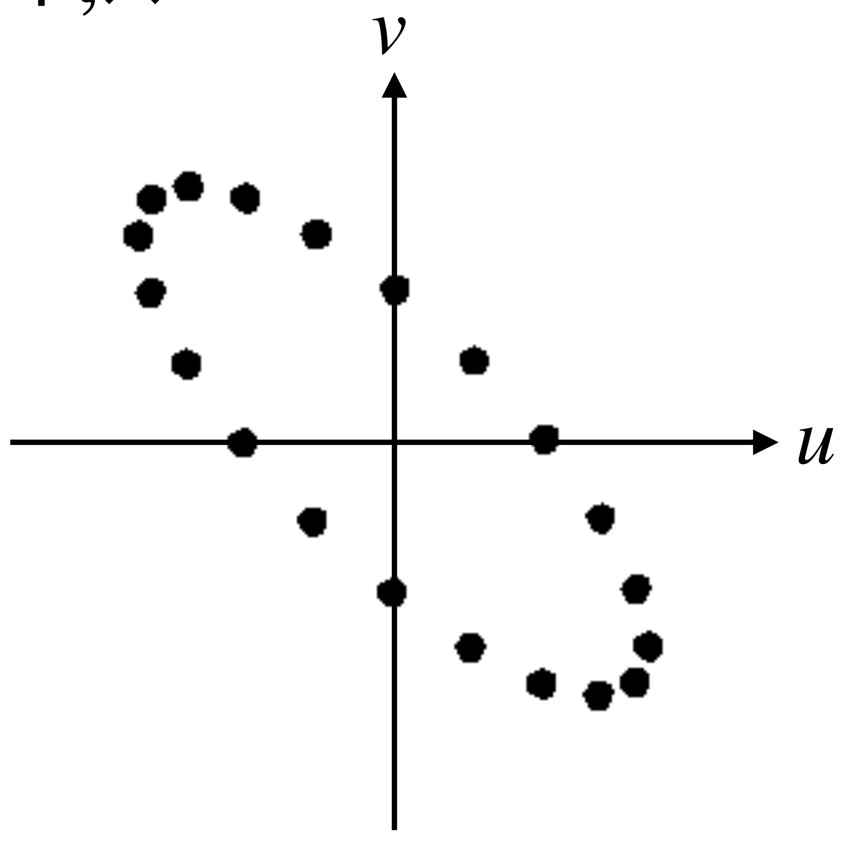


# Polarization of GWs

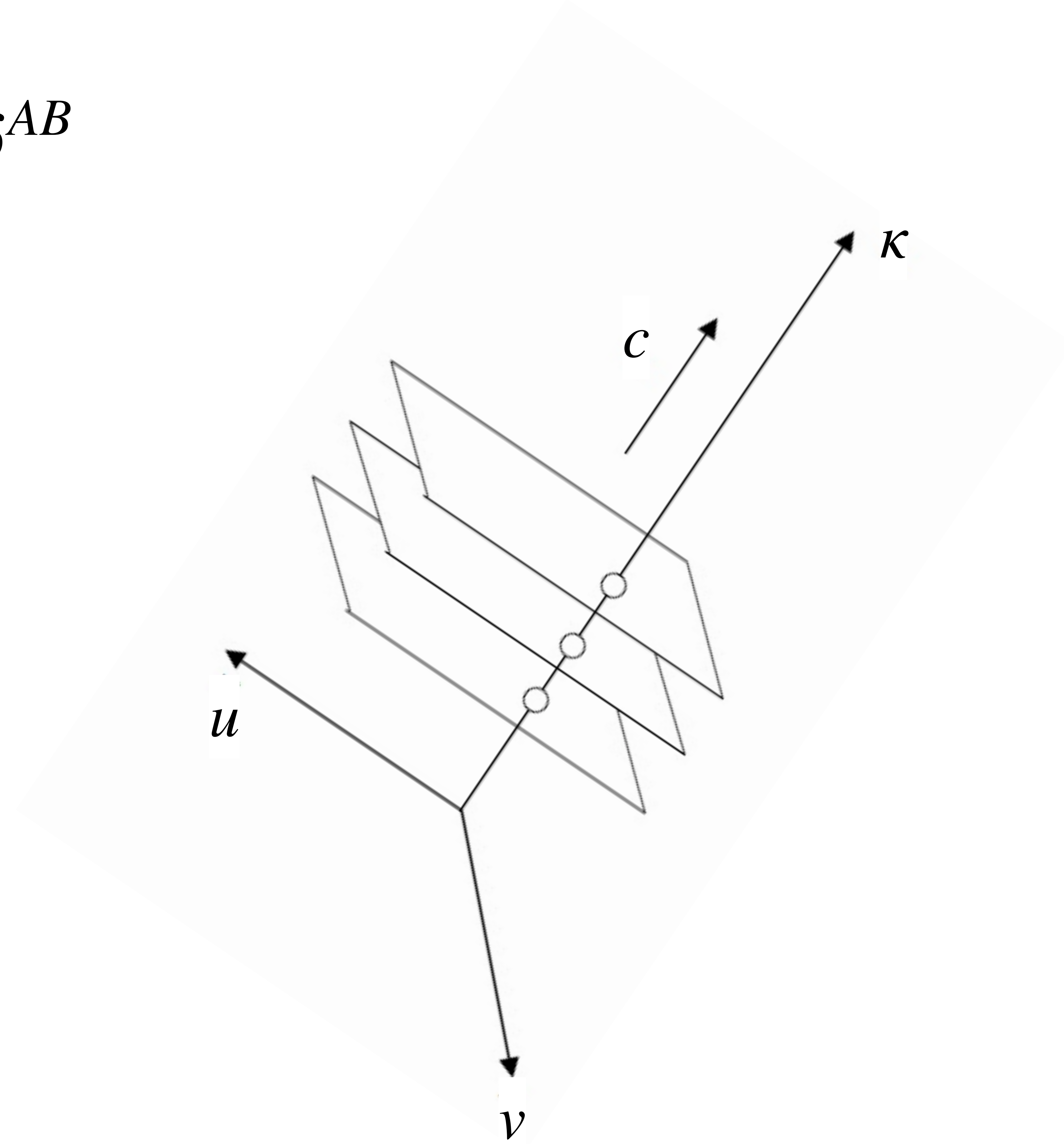
- $e^{+}_{ab} = \frac{1}{\sqrt{2}}\left(u_a u_b - v_a v_b\right)$ s.t. $e^{A}_{ab} e^{B}_{cd} g^{ac} g^{bd} = \delta^{AB}$
- $e^{\times}_{ab} = \frac{1}{\sqrt{2}}\left(u_a v_b + v_a u_b\right)$
- $\tilde{h}_{ab} = \tilde{h}_{+} e^{+}_{ab} + \tilde{h}_{\times} e^{\times}_{ab} = \sum_{A=+,\times} \tilde{h}_A e^{A}_{ab} = \tilde{h}_A e^{A}_{ab}$




+ polarization

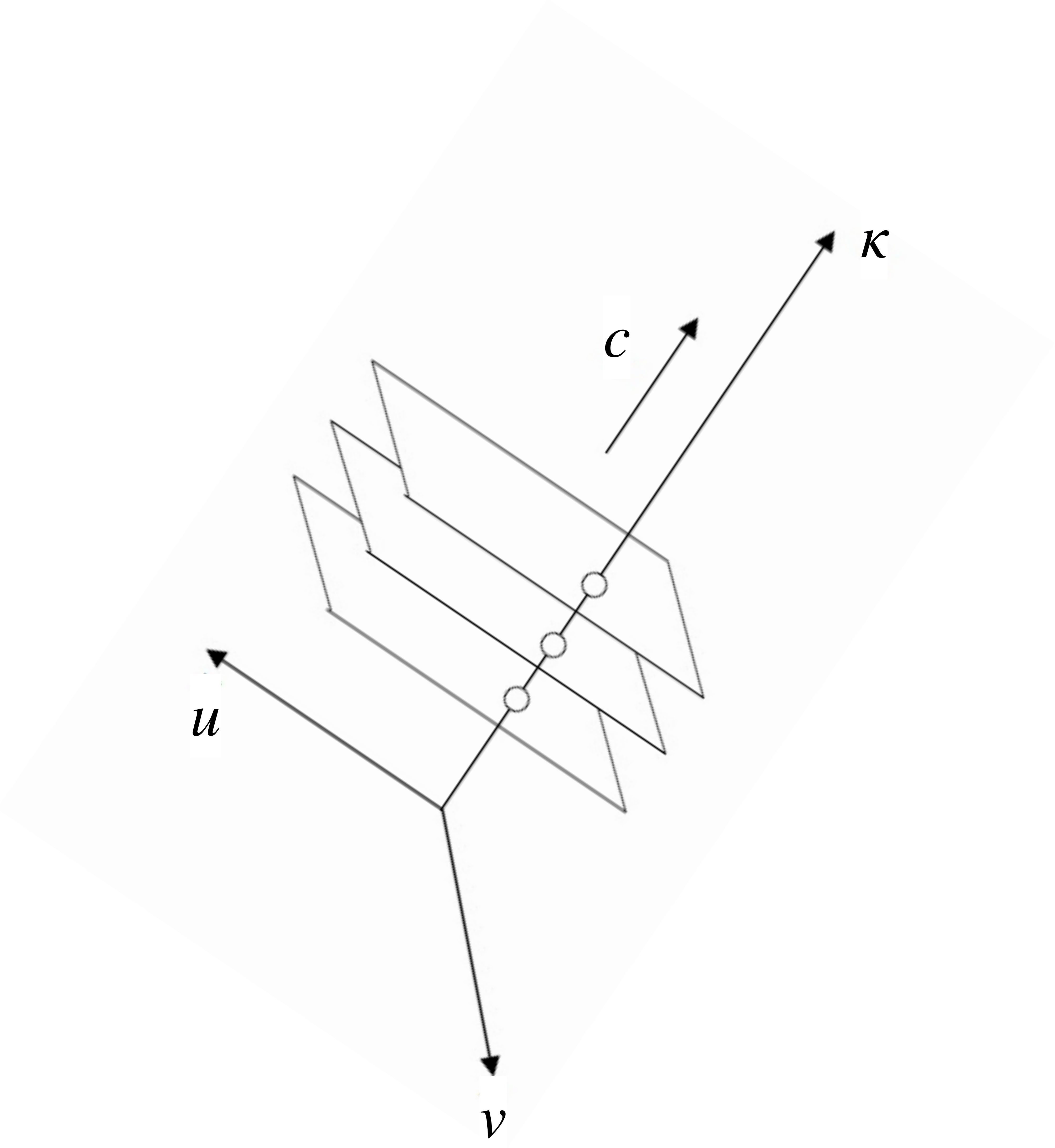



× polarization



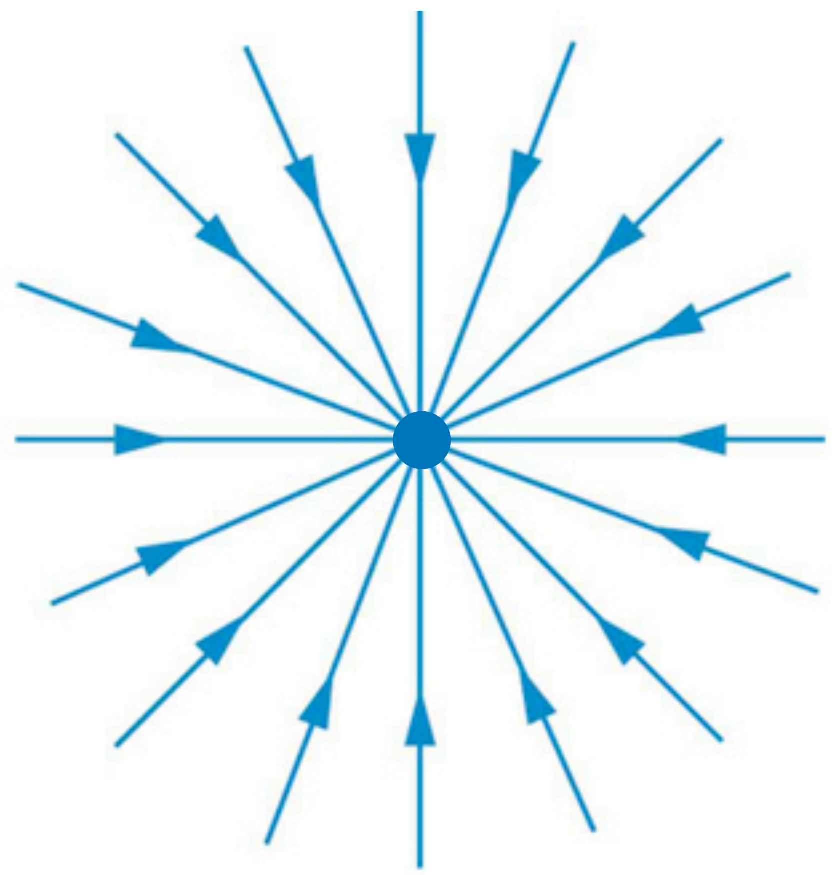
# Plane GWs

- $h_{ab}\left(t,\vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\tilde{h}_{ab}(\omega)\,e^{iP\left(t,\vec{x};\,\omega\right)}$
- Complex amplitude: $\tilde{h}_{ab}(\omega) = \tilde{h}_A(\omega)\,e^A_{ab}$
- Note that $\tilde{h}_{ab}(-\omega) = \tilde{h}^*_{ab}(\omega)$
- Phase: $P\left(t,\vec{x};\omega\right) = \omega\left(-t + \kappa\cdot\vec{x}\right)$



# General GWs

- $h_{ab}\left(t,\vec{x}\right) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{h}_{ab}(\omega,\kappa)\, e^{iP\left(t,\vec{x};\,\omega,\kappa\right)}$
- Integration over all directions: $\int d^2\kappa = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta$
  - where $\kappa = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$
- Complex amplitude: $\tilde{h}_{ab}(\omega,\kappa) = \tilde{h}_A(\omega,\kappa)\, e^A_{ab}(\kappa)$
- Note that $\tilde{h}_{ab}(-\omega,\kappa) = \tilde{h}^*_{ab}(\omega,\kappa)$
- Phase: $P\left(t,\vec{x};\omega,\kappa\right) = \omega\left(-t + \kappa\cdot\vec{x}\right)$
- All information of GWs are encoded in $\tilde{h}_{ab}(\omega,\kappa)$.



All directions

# Stochastic GWs

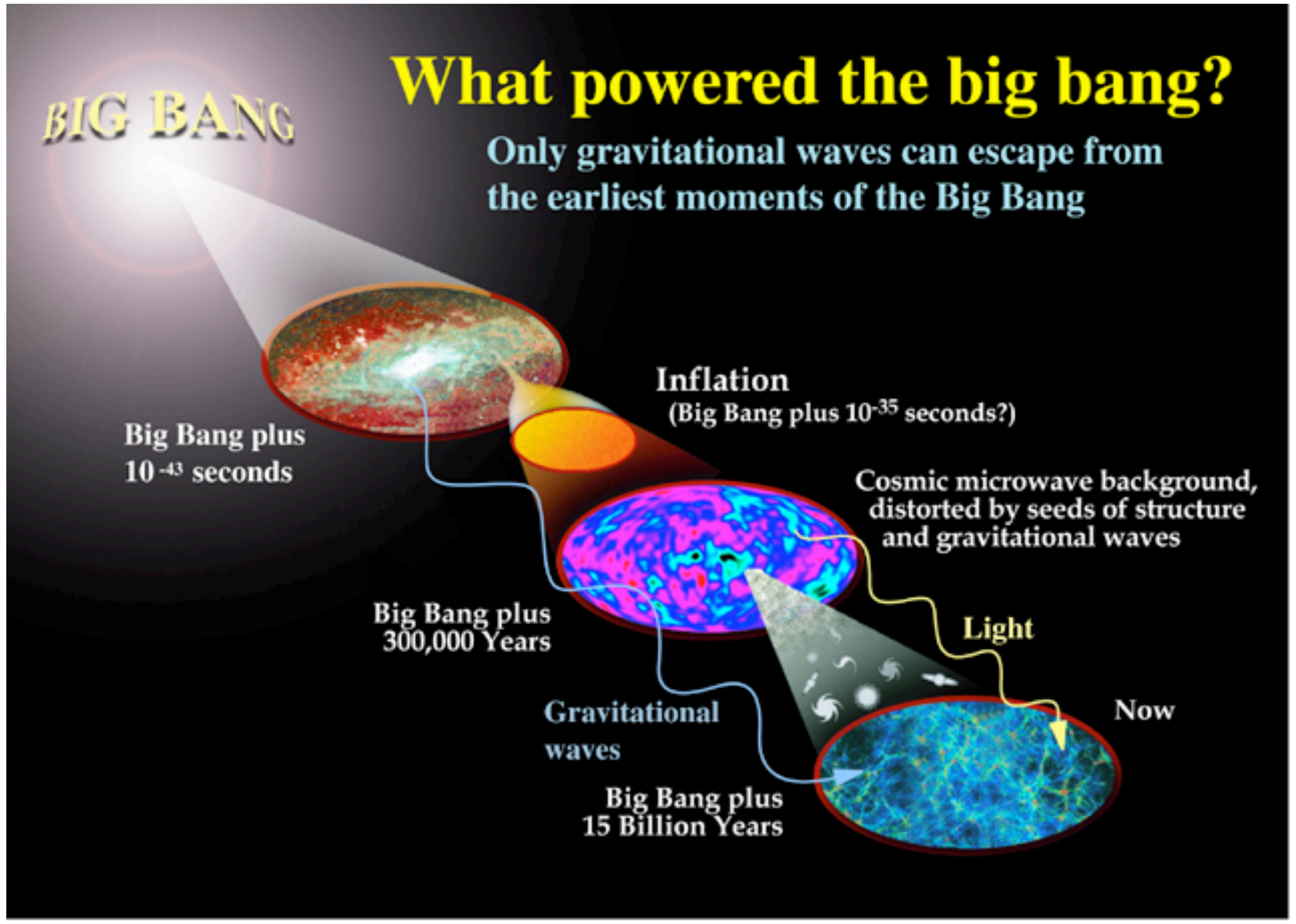
- $h_{ab}(t,\vec{x}) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{h}_{ab}(\omega,\kappa)\, e^{iP(t,\vec{x};\,\omega,\kappa)}$
- $\tilde{h}_{ab}(\omega,\kappa)$ is statistical random variable.
- First moment of random variable
  - $\left\langle \tilde{h}_{ab}(\omega,\kappa) \right\rangle = ?$
- Second moment of random variable
  - $\left\langle \tilde{h}_{ab}(\omega,\kappa)\, \tilde{h}^*_{cd}(\omega',\kappa') \right\rangle = ?$


An example of stochastic process

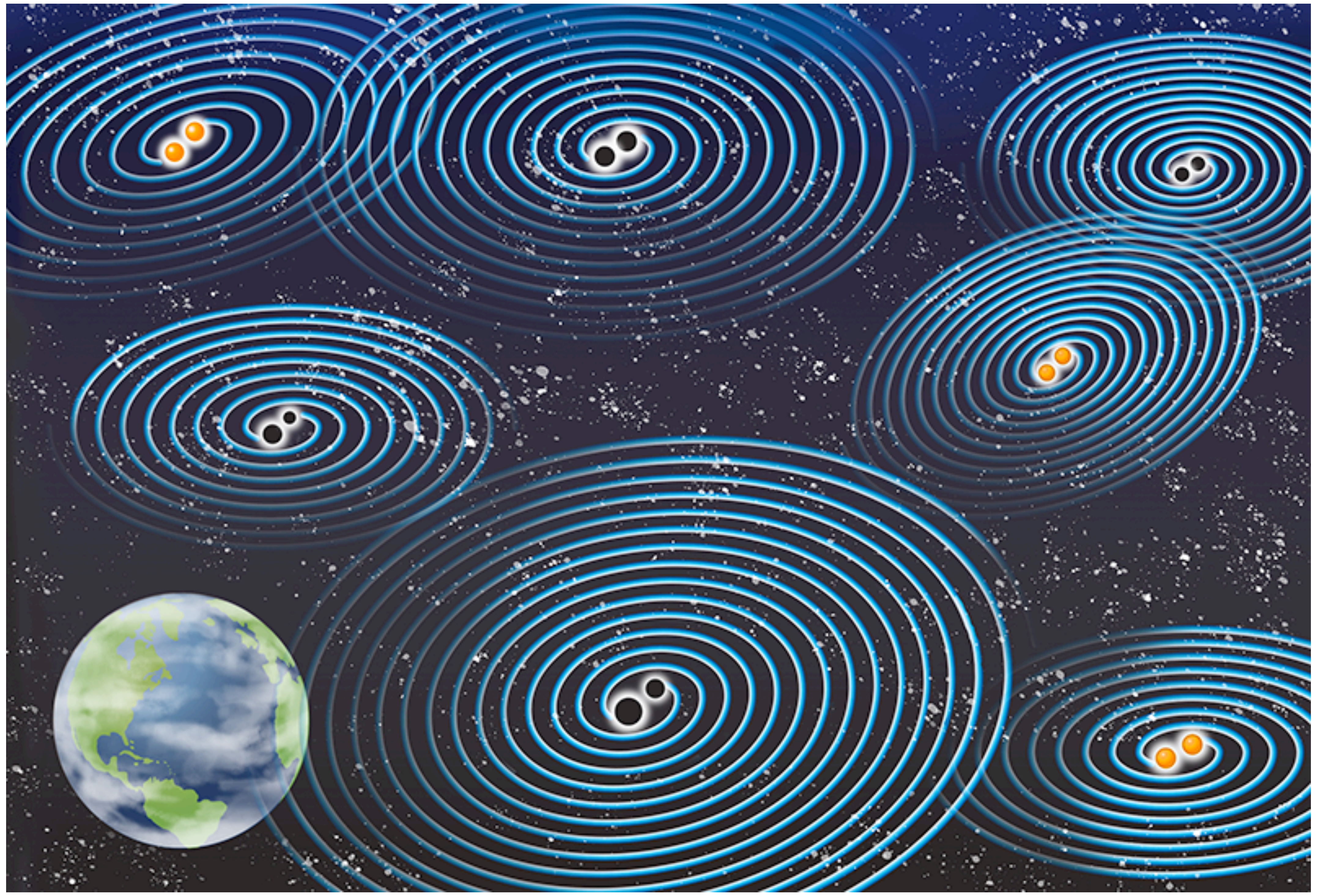
# Stochastic Gravitational Wave Background

# Stochastic Gravitational Wave Background

- Cosmological origin: Quantum state in early universe
- Astrophysical origin: Distribution of compact binaries




Stochastic GW from Big Bang / Credit: NASA



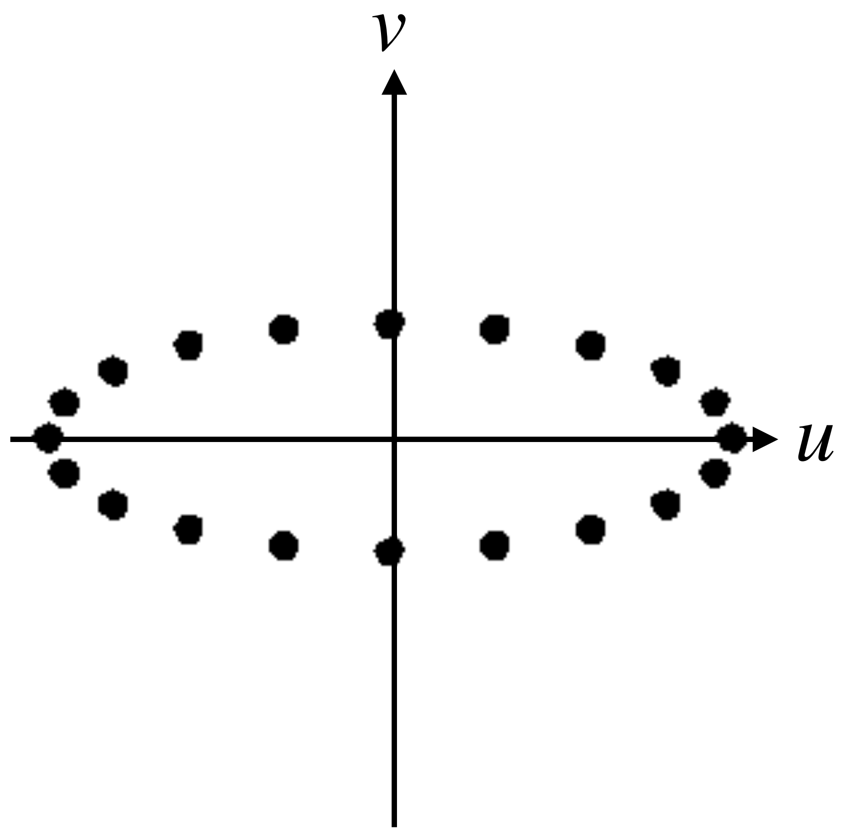
Distribution of Compact Binaries / Credit: APS

# Statistical Assumptions on SGWB

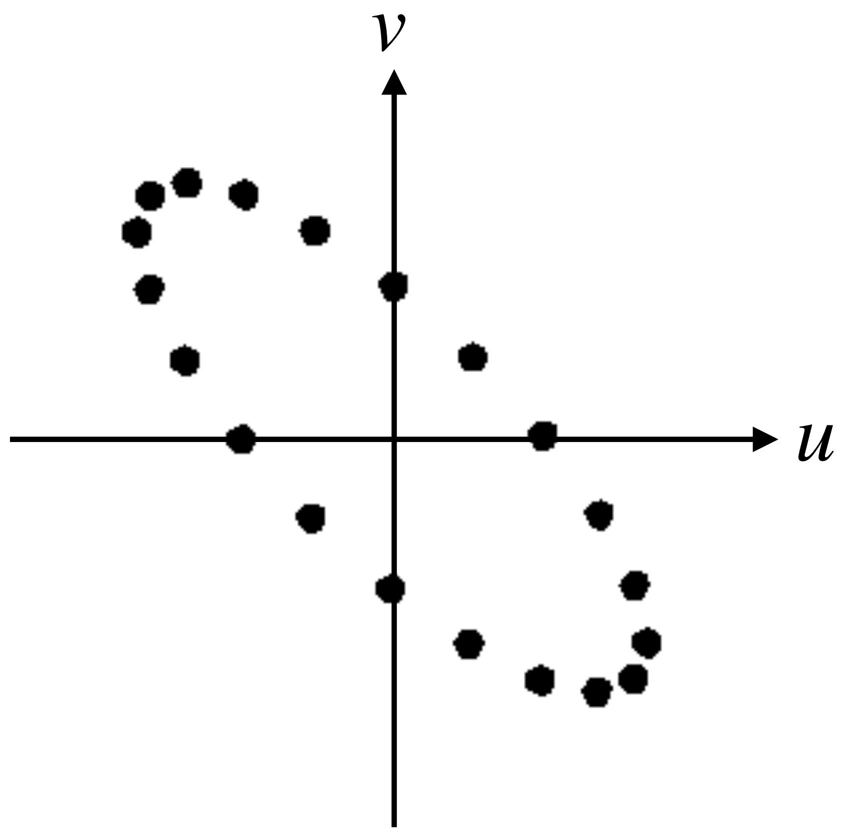
- Gaussian and stationary assumptions
  - $\left\langle \tilde{h}_{ab}(\omega, \kappa) \right\rangle = 0$
  - $\left\langle \tilde{h}_{ab}(\omega, \kappa)\, \tilde{h}^*_{cd}(\omega', \kappa') \right\rangle \propto S_h(\omega)\, \delta(\omega - \omega')$
  - $S_h(\omega)$: power spectral density (real and even)
- Isotropic assumption
  - $\left\langle \tilde{h}_{ab}(\omega, \kappa)\, \tilde{h}^*_{cd}(\omega', \kappa') \right\rangle \propto \delta^2(\kappa - \kappa')$
  - $\delta^2(\kappa - \kappa') = \delta(\cos\theta - \cos\theta')\, \delta\left(\phi - \phi'\right)$
  - $\kappa = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$

# Statistical Assumptions on SGWB

- No prefer polarization assumption
  - $\left\langle \tilde{h}_{ab}(\omega,\kappa)\,\tilde{h}^{*}_{cd}(\omega',\kappa')\right\rangle \propto \Lambda_{abcd}(\kappa)$
  - $\Lambda^{ab}{}_{cd} = e^{ab}_{A}e^{A}_{cd} = P^{a}{}_{(c}P^{b}{}_{d)} - \frac{1}{2}P^{ab}P_{cd}$: projection operator for symmetric traceless rank-2 tensors in $u-v$ plane
  - $P^{a}{}_{b} = \delta^{a}{}_{b} + n^{a}n_{b} - \kappa^{a}\kappa_{b}$: projection operator for vector to $u-v$ plane




+ polarization




× polarization

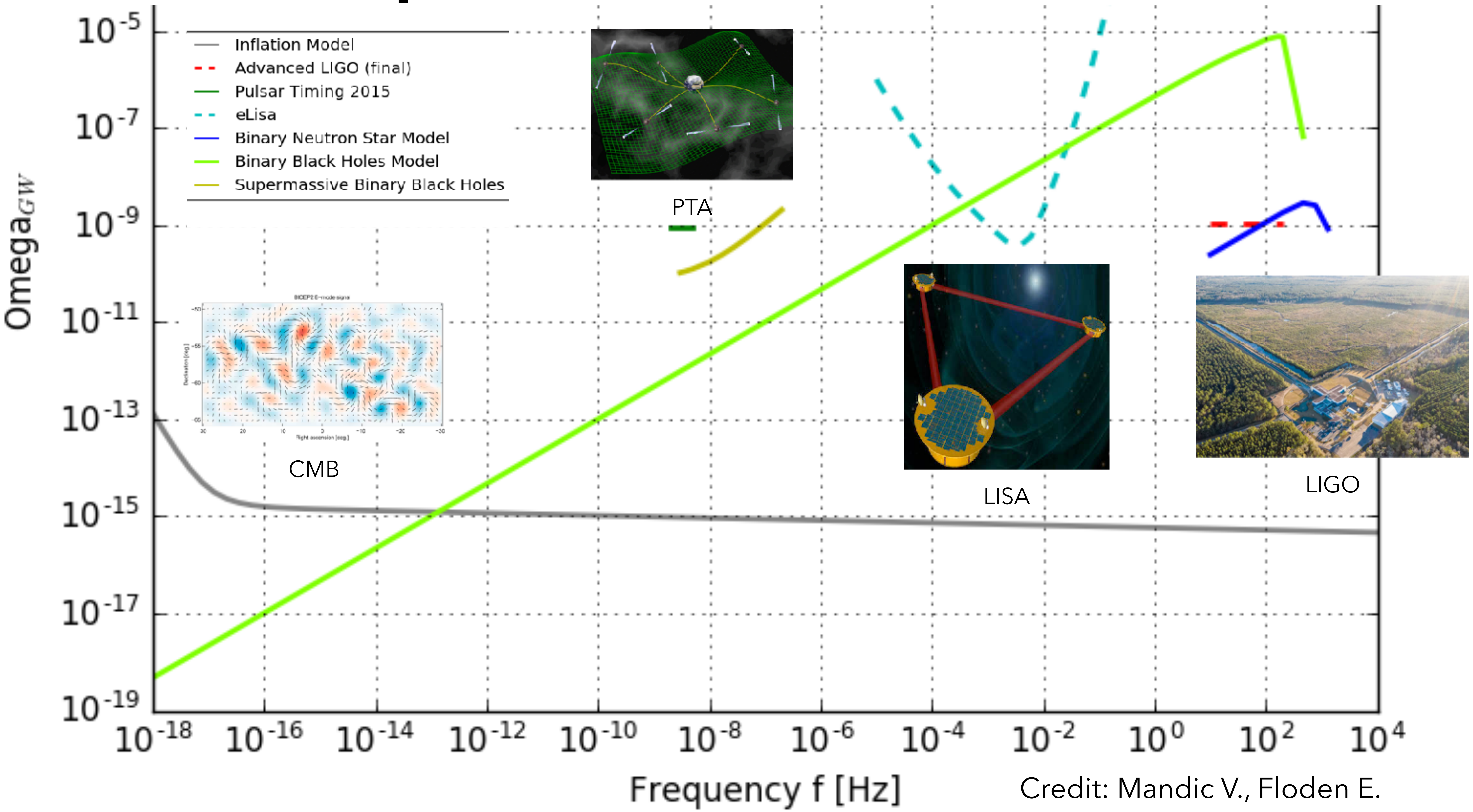
# Second Moment of SGWBs

- $\left\langle \tilde{h}_{ab}(\omega, \kappa)\, \tilde{h}^{*}_{cd}(\omega', \kappa') \right\rangle = 2\pi\delta(\omega - \omega')\, S_h(\omega)\, \frac{1}{4\pi}\delta^2(\kappa - \kappa')\, \frac{1}{2}\Lambda_{abcd}(\kappa)$
- All information of SGWB are encoded in $S_h(\omega)$
- $$\begin{aligned} \left\langle h_{ab}h^{ab} \right\rangle &= \int d^2\kappa \int d^2\kappa' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \tilde{h}_{ab}(\omega, \kappa)\, \tilde{h}^{ab*}(\omega', \kappa') \right\rangle e^{iP\left(t, \vec{x}; \omega, \kappa\right)} e^{-iP\left(t, \vec{x}; \omega', \kappa'\right)} \\ &= \frac{1}{4\pi} \int d^2\kappa \int d^2\kappa' \delta^2(\kappa - \kappa') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)\, e^{i\omega\left(-t + \kappa\cdot\vec{x}\right)} e^{-i\omega\left(-t + \kappa'\cdot\vec{x}\right)} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega) \end{aligned}$$
- It justifies the name of $S_h(\omega)$, i.e., the power spectral density.
- In usual text, the result has factor of 4 due to the careless normalization.

# "Omega GW" of SGWB

- Energy density of SGWB
  - $\rho_{gw} = \frac{1}{32\pi} \left\langle \partial_t h_{ab} \partial_t h^{ab} \right\rangle = \frac{1}{32\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 S_h(\omega) = \frac{1}{32\pi} \int_0^{\infty} df \left(2\pi f\right)^2 S_h^{\text{one-sided}}\left(f\right)$
- Omega GW
  - $\Omega_{\text{gw}} = \frac{d\left(\rho_{gw}/\rho_c\right)}{d \ln f} = \frac{\pi^2}{3H_0^2} f^3 S_h^{\text{one-sided}}\left(f\right)$
  - where $\rho_c = \frac{3H_0^2}{8\pi}$

# Spectrum of SGWBs




Credit: Mandic V., Floden E.

# Noise Reduction: Two-Detector Correlation Method

# Output vs Signal

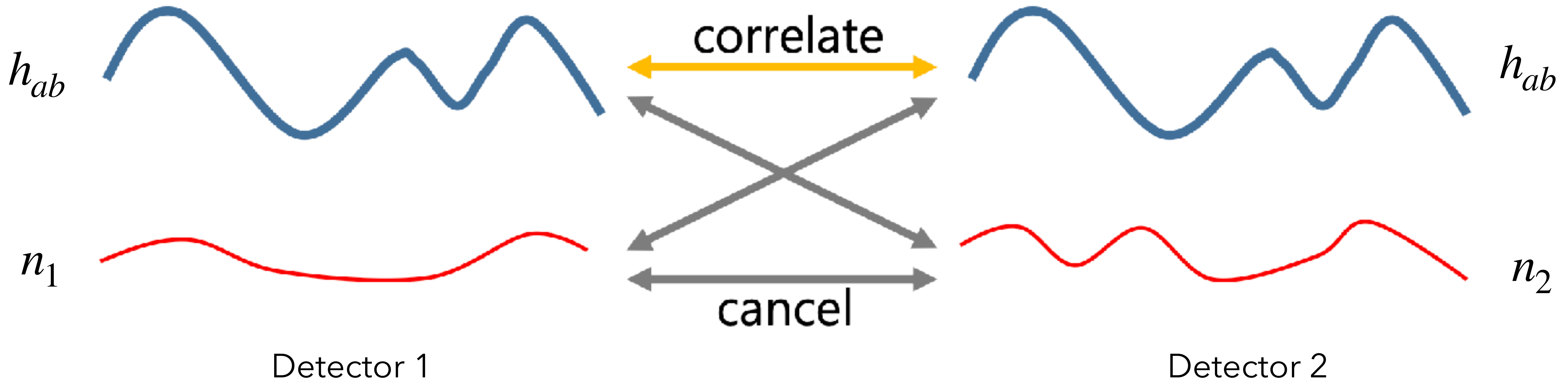
- $s(t) = h(t) + n(t)$
- $$h(t) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{D}^{ab}(\omega)\, \tilde{h}_{ab}(\omega, \kappa)\, e^{i\omega\left(-t + \kappa \cdot \vec{x}_0\right)}$$
$$= \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{D}^{ab}(\omega)\, e^{A}_{ab}(\kappa)\, \tilde{h}_A(\omega, \kappa)\, e^{i\omega\left(-t + \kappa \cdot \vec{x}_0\right)}$$
$$= \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{F}^{A}(\omega, \kappa)\, \tilde{h}_A(\omega, \kappa)\, e^{i\omega\left(-t + \kappa \cdot \vec{x}_0\right)}$$
- $\tilde{D}^{ab}$: detector tensor
- $\tilde{F}^{A}$: pattern function
- $\tilde{h}(\omega) = \int d^2\kappa \tilde{D}^{ab}(\omega)\, \tilde{h}_{ab}(\omega, \kappa)\, e^{i\kappa\cdot\vec{x}_0}$

# Noise

- $n(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{n}(\omega) e^{-i\omega t}$
- Gaussian and stationary assumptions
  - $\langle \tilde{n}(\omega) \rangle = 0$
  - $\langle \tilde{n}(\omega) \tilde{n}^*(\omega') \rangle = 2\pi\delta(\omega - \omega') S_n(\omega)$
  - $S_n(\omega)$: noise spectral density (real and even)
- All information of the noise are encoded in $S_n(\omega)$
- $\left\langle n^2(t) \right\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_n(\omega)$

# Correlation Method

- Output for two-detectors
  - $s_1(t) = h_1(t) + n_1(t) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{F}_1^A(\omega, \kappa)\, \tilde{h}_A(\omega, \kappa)\, e^{i(-t + \kappa \cdot \vec{x}_1)} + n_1(t)$
  - $s_2(t) = h_2(t) + n_2(t) = \int d^2\kappa \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{F}_2^A(\omega, \kappa)\, \tilde{h}_A(\omega, \kappa)\, e^{i(-t + \kappa \cdot \vec{x}_2)} + n_2(t)$



# Correlation Measure

- $$Y = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, s_1(t)\, s_2(t')\, Q(t - t')$$
  $$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi}\, \tilde{s}_1(\omega)\, \tilde{s}_2^*(\omega')\, \tilde{Q}(\omega'')\, 2\pi\delta_T(\omega - \omega')\, 2\pi\delta_T(\omega' - \omega'')$$
  $$\simeq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{s}_1(\omega)\, \tilde{s}_2^*(\omega)\, \tilde{Q}(\omega)$$
- $Q(t)$: real filter function
- $\delta_T(\omega) = \dfrac{\sin(\omega T/2)}{\pi\omega} \to \delta(\omega)$ as $\omega T \to \infty$
- $\lim_{\omega\to 0} \delta_T(\omega) = T/2\pi$

# Signal to Noise Ratio

Remark that $\left\langle \tilde{h}_{ab}(\omega,\kappa)\,\tilde{h}^*_{cd}(\omega',\kappa')\right\rangle = 2\pi\delta(\omega-\omega')\,S_h(\omega)\,\frac{1}{4\pi}\delta^2(\kappa-\kappa')\,\frac{1}{2}\Lambda_{abcd}(\kappa)$

- Signal: Expectation of $Y$
- $S = \langle Y \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\langle \tilde{h}_1(\omega)\,\tilde{h}_2^*(\omega)\right\rangle \tilde{Q}(\omega)$

$$\left\langle \tilde{h}_1(\omega)\,\tilde{h}_2^*(\omega)\right\rangle = \int d^2\kappa \int d^2\kappa' \tilde{D}_1^{ab}(\omega)\,\tilde{D}_2^{*cd}(\omega)\left\langle \tilde{h}_{ab}(\omega,\kappa)\,\tilde{h}^*_{cd}(\omega,\kappa')\right\rangle e^{i\omega\kappa\cdot\vec{x}_1}e^{-i\omega\kappa'\cdot\vec{x}_2}$$

- $$\simeq 2\pi\delta_T(0)\,S_h(\omega)\,\frac{1}{2}\tilde{D}_1^{ab}(\omega)\,\tilde{D}_2^{*cd}(\omega)\,\frac{1}{4\pi}\int d^2\kappa\,\Lambda_{abcd}(\kappa)\,e^{i\omega\kappa\cdot(\vec{x}_1-\vec{x}_2)}$$

$$= TS_h(\omega)\,\tilde{\Gamma}(\omega)$$

- $\tilde{\Gamma}(\omega) = \frac{1}{2}\tilde{D}_1^{ab}(\omega)\,\tilde{D}_2^{*cd}(\omega)\,\frac{1}{4\pi}\int d^2\kappa\,\Lambda_{abcd}(\kappa)\,e^{i\omega\kappa\cdot(\vec{x}_1-\vec{x}_2)}$: overlap reduction function
- $S = T\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)\,\tilde{\Gamma}(\omega)\,\tilde{Q}(\omega)$

# Signal to Noise Ratio

- Noise: Standard deviation of $Y$ without GWs

$$N^2 = \left[\left\langle Y^2 \right\rangle - \langle Y \rangle^2\right]_{h=0}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_2^*(\omega)\, \tilde{n}_1^*(\omega')\, \tilde{n}_2(\omega') \right\rangle \tilde{Q}(\omega)\, \tilde{Q}^*(\omega') + \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_2^*(\omega) \right\rangle \tilde{Q}(\omega)\right]^2$$

- $$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \tilde{n}_1(\omega)\, \tilde{n}_1^*(\omega') \right\rangle \left\langle \tilde{n}_2(\omega')\, \tilde{n}_2^*(\omega) \right\rangle \tilde{Q}(\omega)\, \tilde{Q}^*(\omega')$$

$$= T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{n,1}(\omega)\, S_{n,2}(\omega) \left|\tilde{Q}(\omega)\right|^2$$

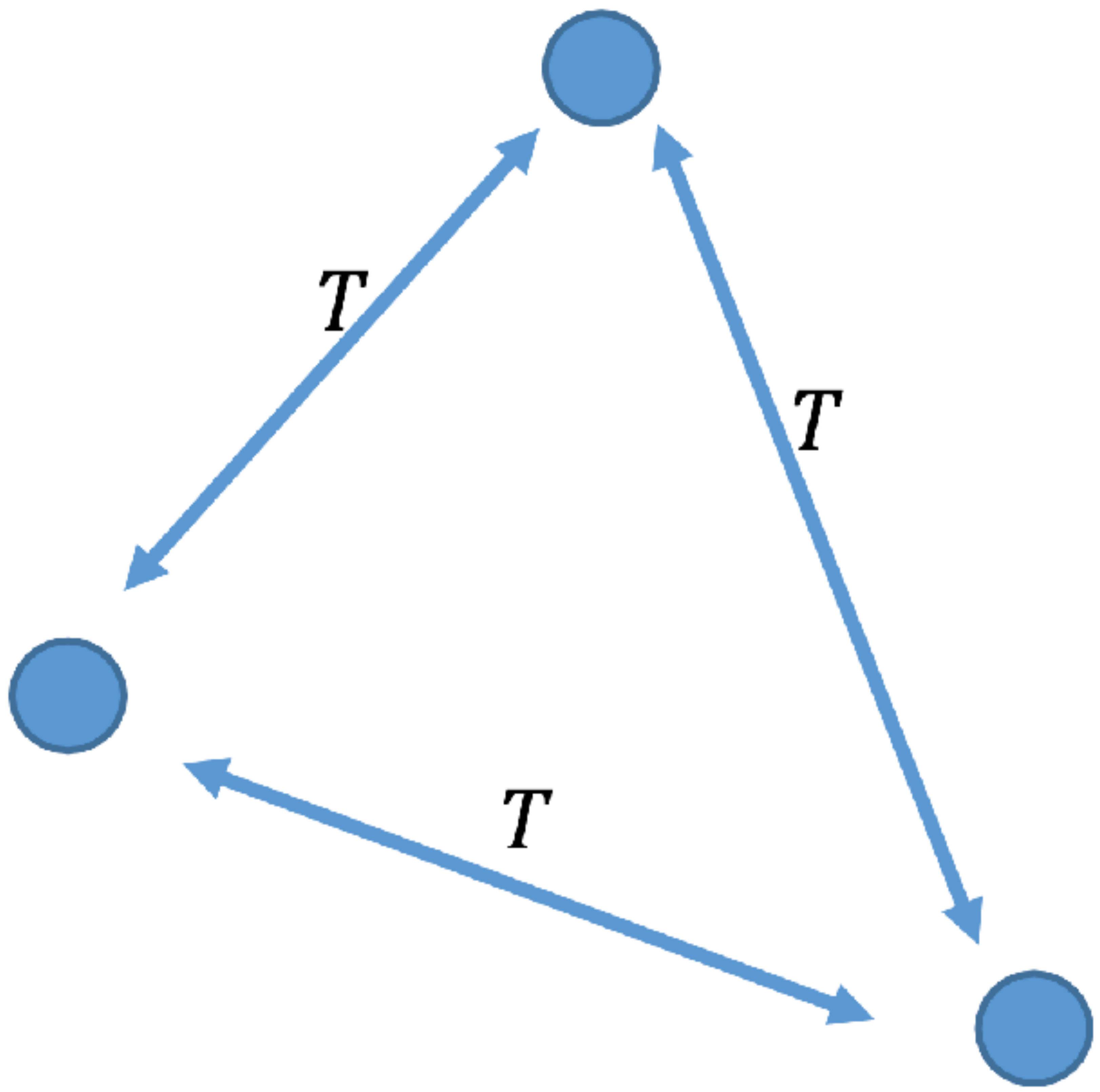
- $$\frac{S}{N} = \sqrt{T} \frac{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)\, \tilde{\Gamma}(\omega)\, \tilde{Q}(\omega)}{\sqrt{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{n,1}(\omega)\, S_{n,2}(\omega) \left|\tilde{Q}(\omega)\right|^2}}$$

# Maximal Signal to Noise Ratio

- We have to determine $Q(t)$ to maximize SNR.
- $$\frac{S}{N} = \sqrt{T}\frac{\left\langle \tilde{Q}, \tilde{\Gamma} S_h / S_{n,1} S_{n,2} \right\rangle}{\sqrt{\left\langle \tilde{Q}, \tilde{Q} \right\rangle}}$$
- where $\left\langle \tilde{A}, \tilde{B} \right\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{A}(\omega) \tilde{B}^*(\omega) S_{n,1}(\omega) S_{n,2}(\omega)$
- SNR is maximized when $\tilde{Q} = \tilde{\Gamma} S_h / S_{1,n} S_{2,n}$
- Then, maximal SNR is
- $$\frac{S}{N} = \sqrt{T}\left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left|\tilde{\Gamma}(\omega)\right|^2 \frac{S_h^2(\omega)}{S_{n,1}(\omega) S_{n,2}(\omega)}\right]^{1/2}$$

# N-detector network

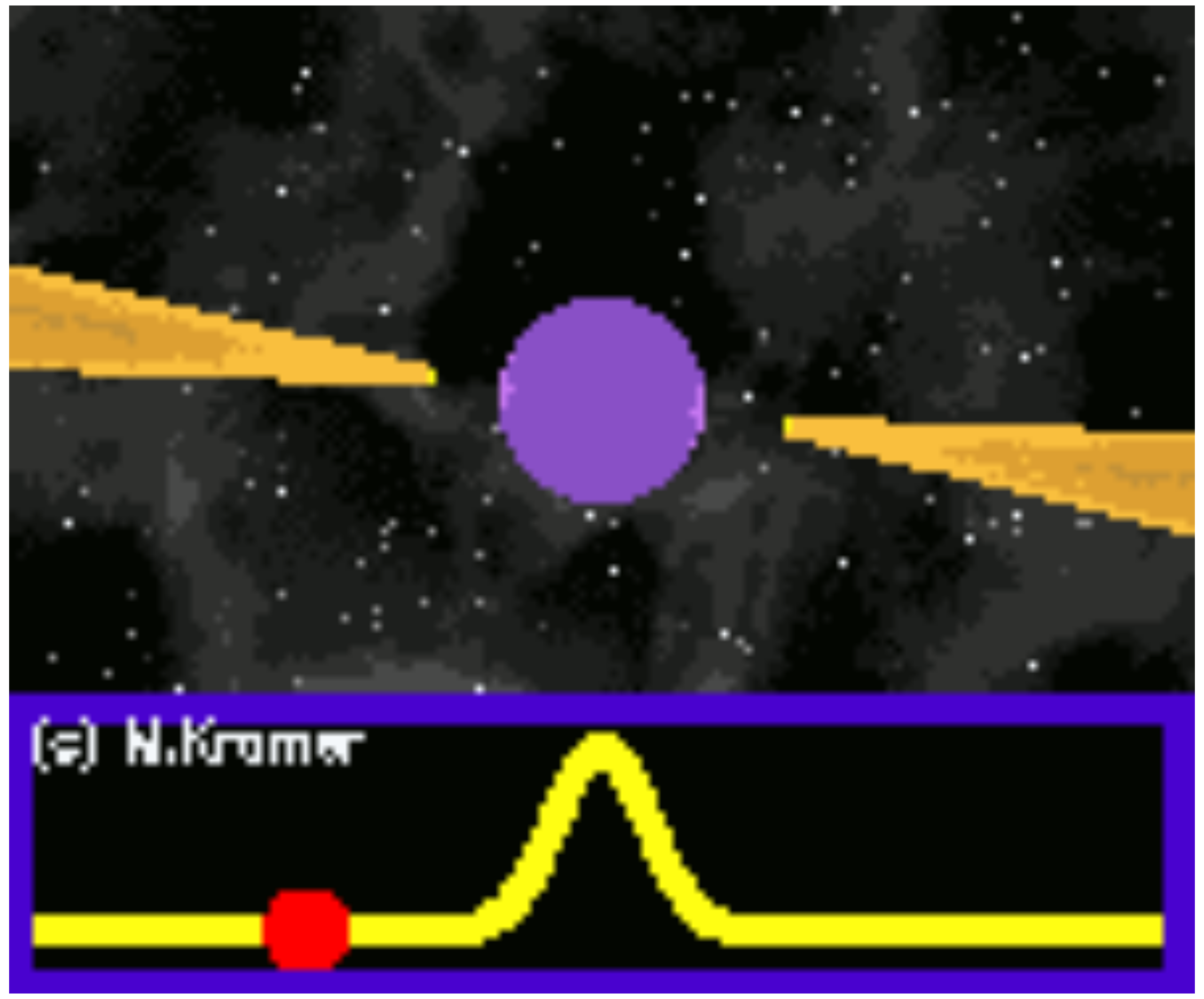
- Total measurement time: $T_N = \frac{N(N-1)}{2} T$



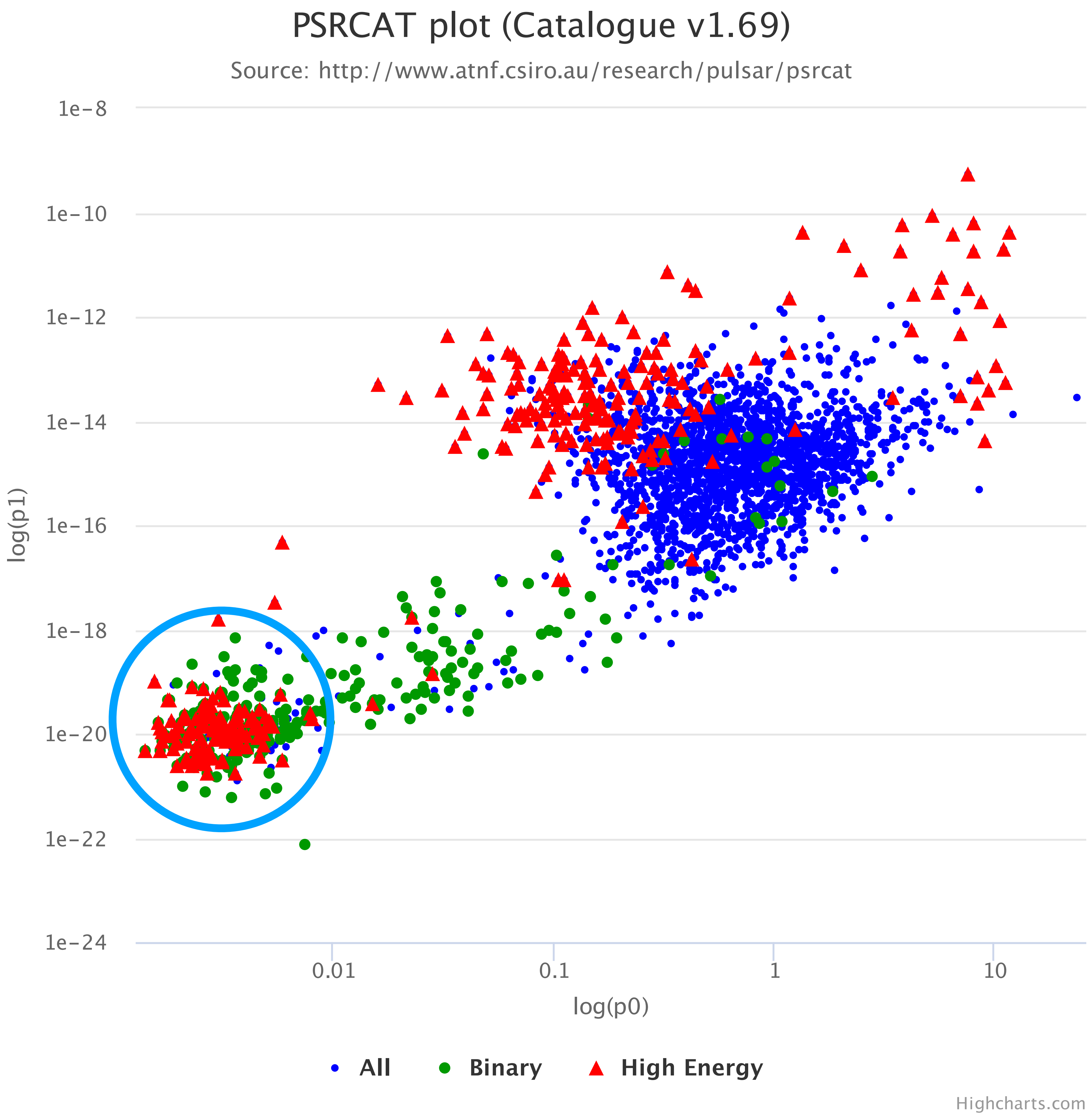
# Pulsar Timing Arrays

# Millisecond Pulsars (MSPs)

- Rotational period $< 10\ \mathrm{ms}$
- Extremely stable rotation
- Usually recycled by a companion star



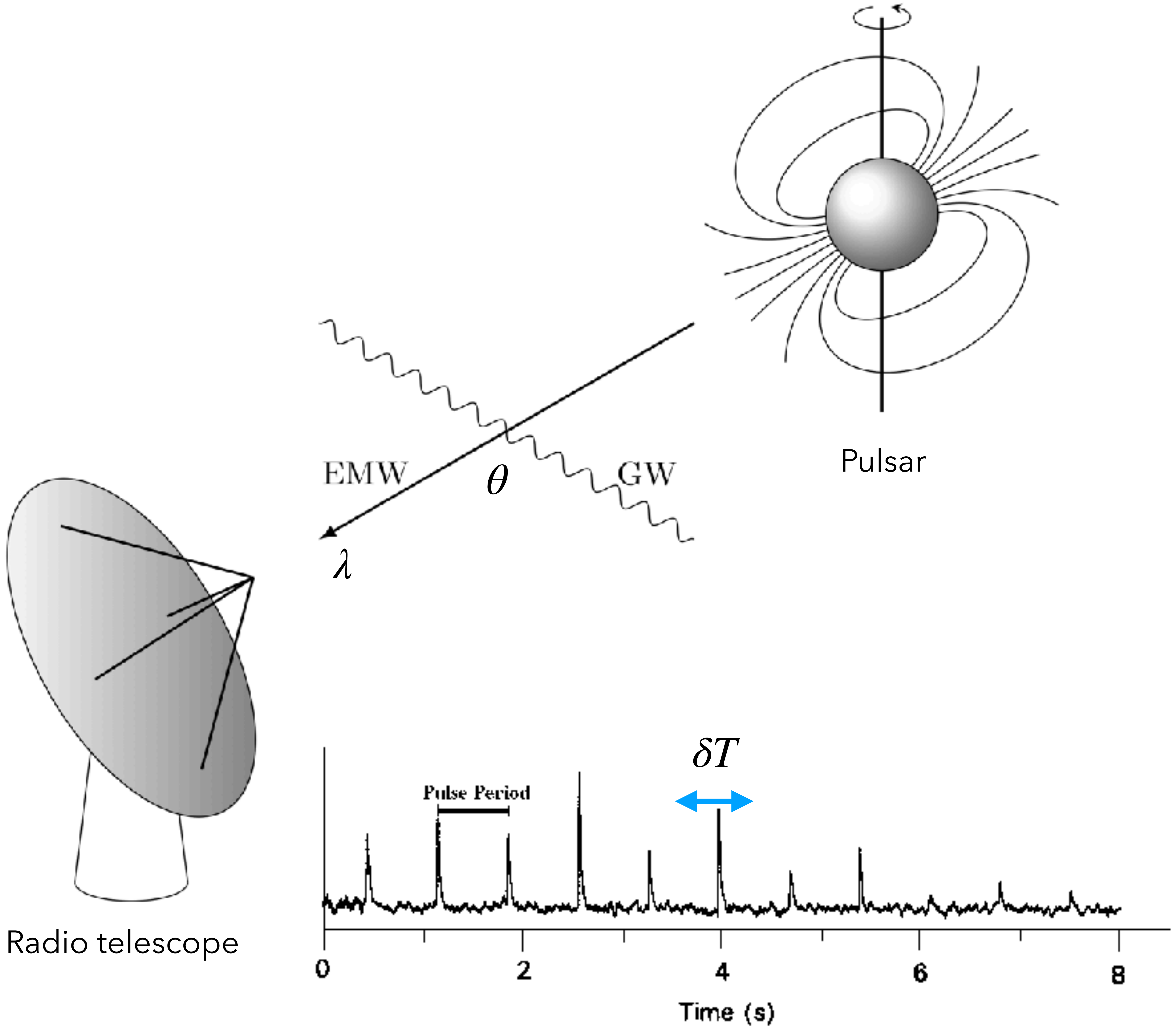

Credit: Michael Kramer (JBCA, University of Manchester).




ATNF Pulsar Catalogue

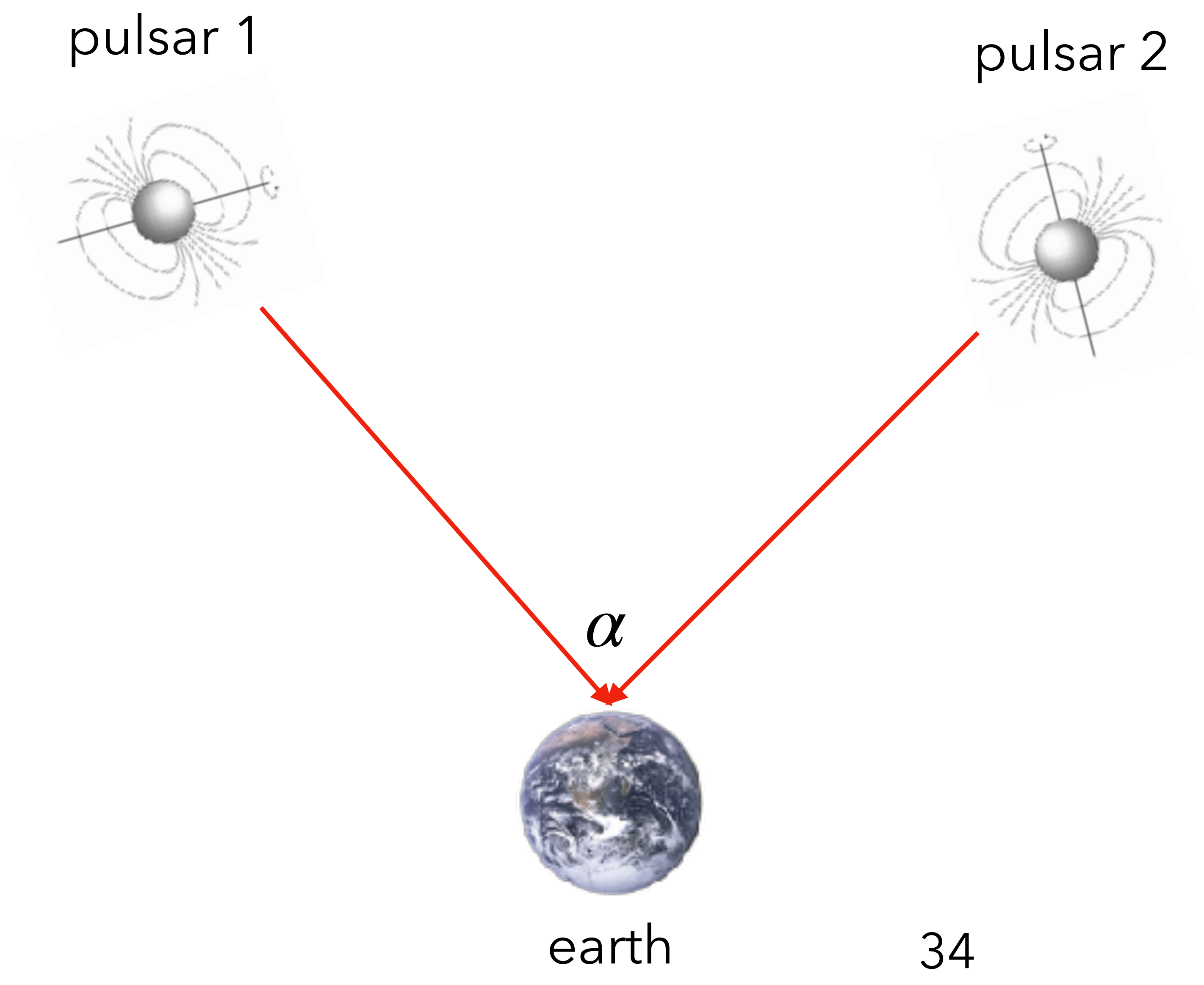
# Perturbation of the Period by GWs

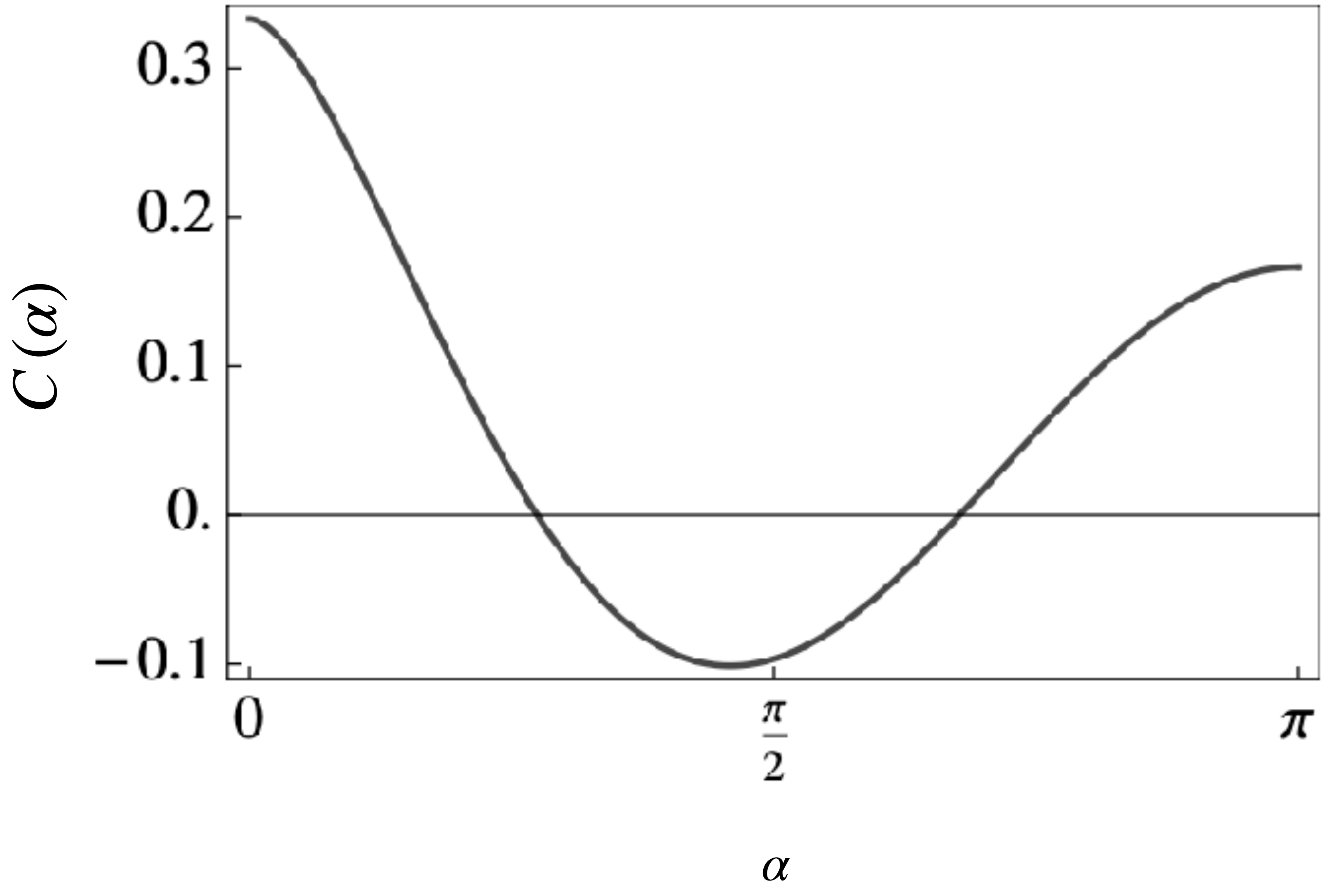
- Electromagnetic waves (EMWs) are perturbed in a spacetime with gravitational waves (GWs).
- Fractional change of the period
  - $\frac{\delta T}{T} = \int d^2\kappa \int \frac{d\omega}{2\pi} \left[ -\frac{\tilde{h}_{ab}\lambda^a\lambda^b}{2\,(1-\cos\theta)} \left(1 - e^{\mathrm{i}\Delta}\right) e^{\mathrm{i}P} \right]$
  - $\tilde{h}$: complex amplitude of GW
  - $\lambda$: spatial unit vector of EMW propagation
  - $\theta$: angle between EMW and GW
  - $\Delta$: time delay phase at pulsar
  - $P$: phase of GW



# Pulsar Timing Array (PTA)

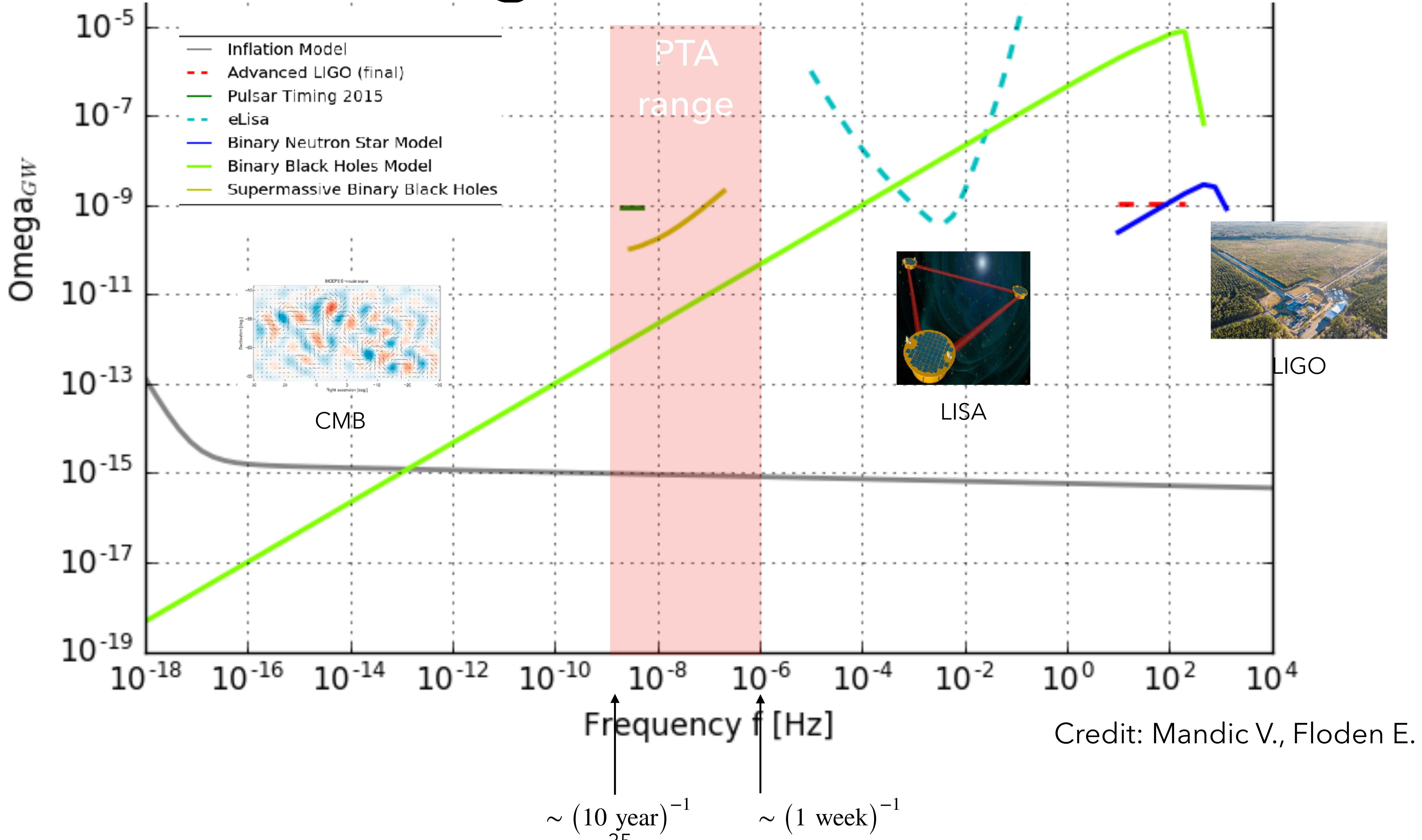
- Power of SGWB
  - $\langle h_{ab} h^{ab} \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega)$
- Correlation of measurements from two pulsars
  - $\left\langle \frac{\delta T_1}{T_1} \frac{\delta T_2}{T_2} \right\rangle = C(\alpha) \langle h_{ab} h^{ab} \rangle$




Hellings and Downs / ApJ (1983)

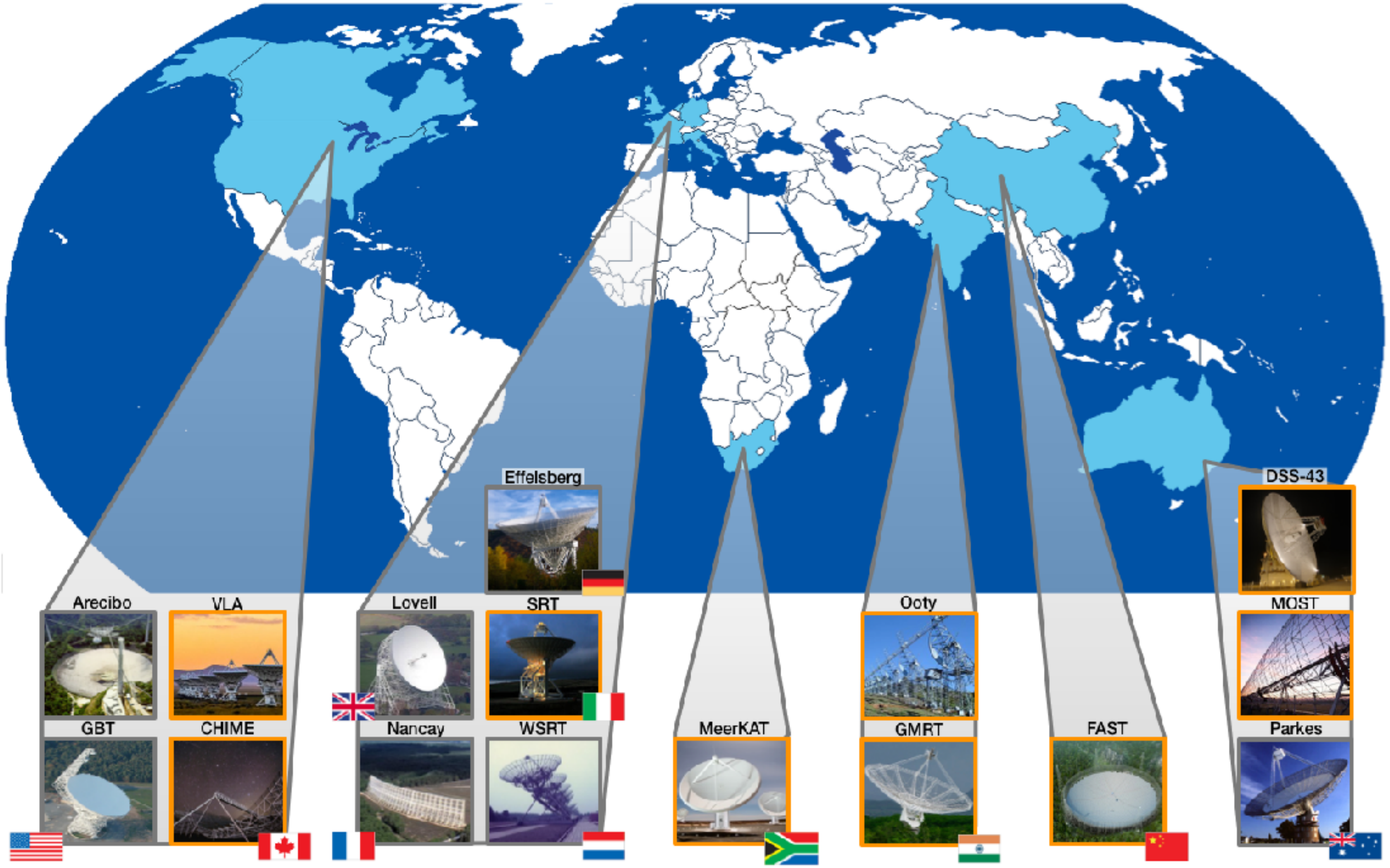
# Target SGWB




Credit: Mandic V., Floden E.

# IPTA

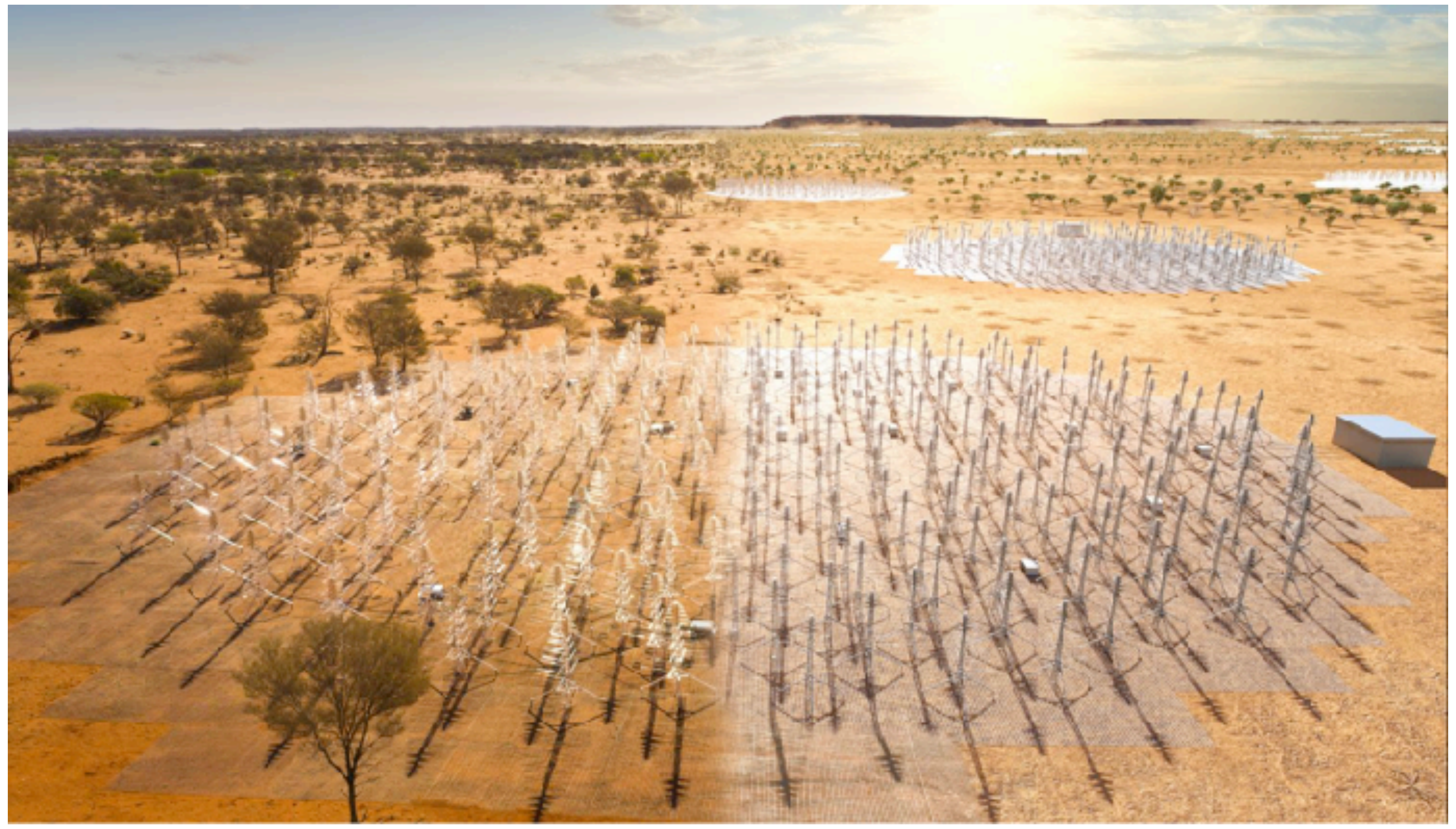
- EPTA+InPTA+NanoGrav+PPTA




Credit: NanoGrav

# Square Kilometer Array Observatory (SKAO)

- SKA Low: 50 - 350 MHz / 131,072 antennas
- SKA Mid: 350 MHz - 15.4 GHz / 197 dishes
- Start from 2028



SKA Low / Australia

SKA Mid / South Africa

# Summary

- SGWB has two origin. (cosmological / astrophysical)
- PTA targets SGWB comes from supermassive binaries with the period ~ 10 years.
- PTAs has accumulated data for ~15 years.
- We will detect SGWB soon.