Effects of eccentricity and aligned spins in the parameter estimation of CBC inspirals

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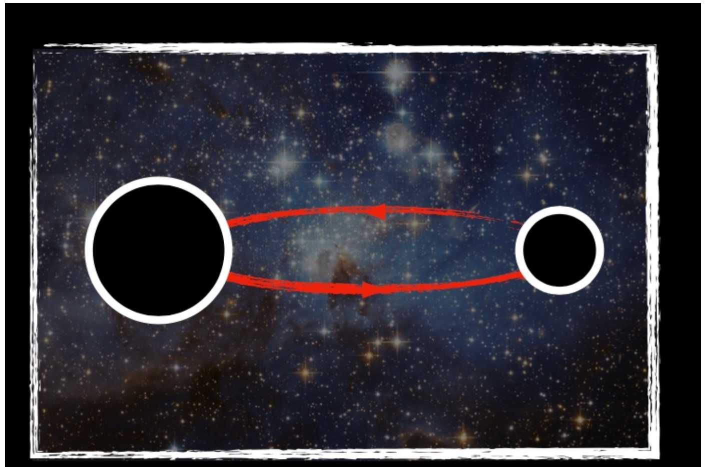
The 69th Workshop on Gravitational Waves and Numerical Relativity @APCTP HQ

Contents

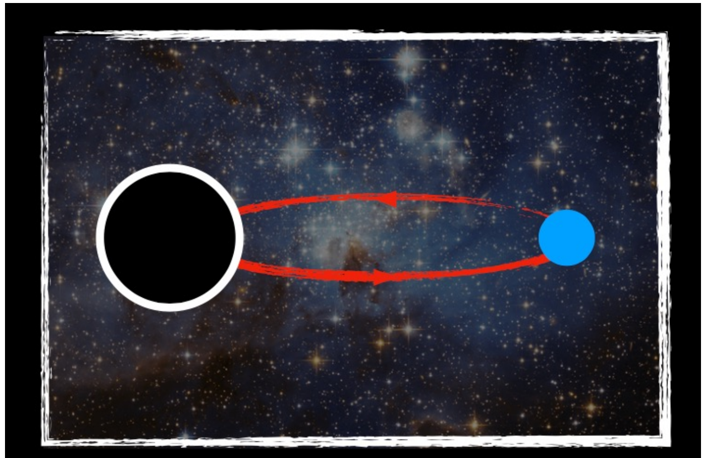
- Gravitational Wave
- Parameter Estimation
  - Bayesian inference
  - MCMC vs Dynesty
- Eccentric GW waveform with aligned-spin

Gravitational Wave : Astrophysical sources

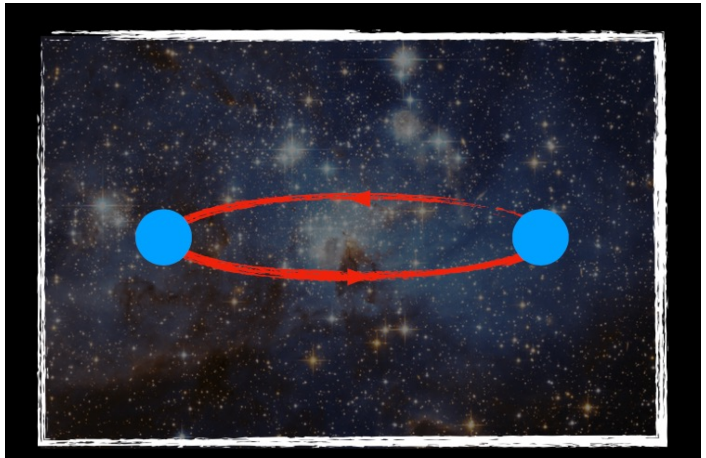
LIGO-Virgo-KAGRA (LVK) focuses on gravitational waves (GWs) from stellar-mass compact binary coalescences (CBCs)



Binary black hole (BBH)



Neutron star-black hole (NSBH)



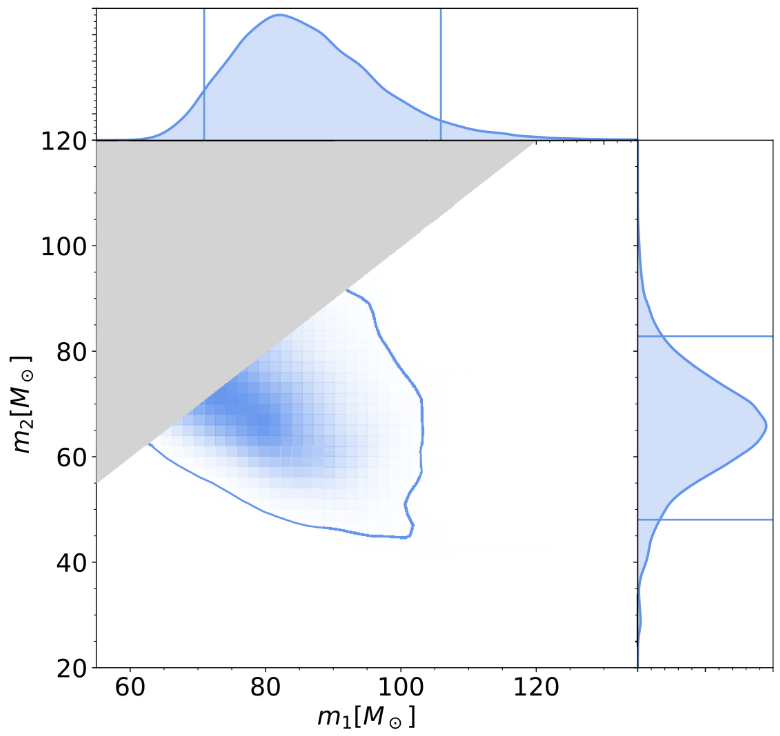
Binary neutron star (BNS)

Other sources exist, e.g. continuous waves (CWs) from rotating neutron stars, or burst waves from core-collapse supernovae.

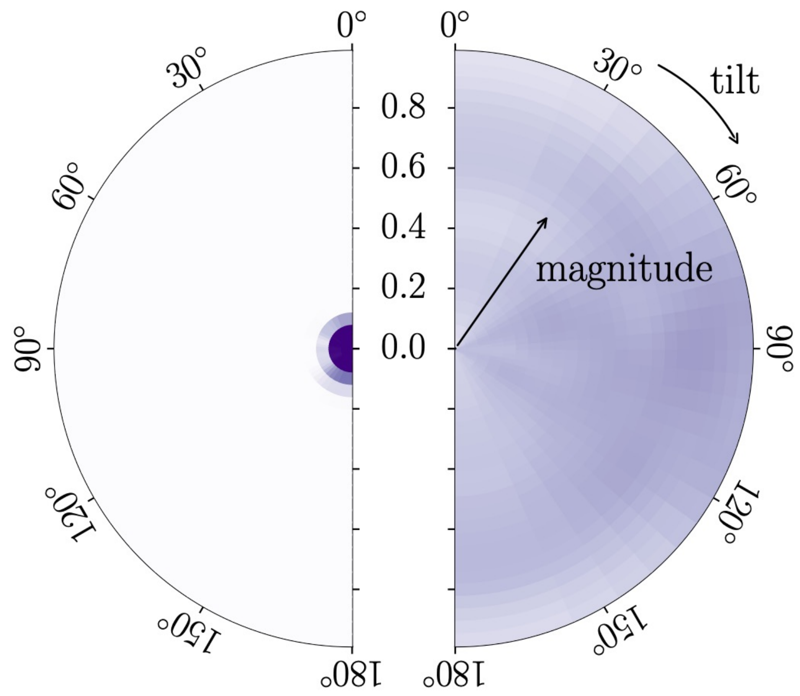
Based GWANW2022 PE tutorial Alan M. Knee

Gravitational Wave

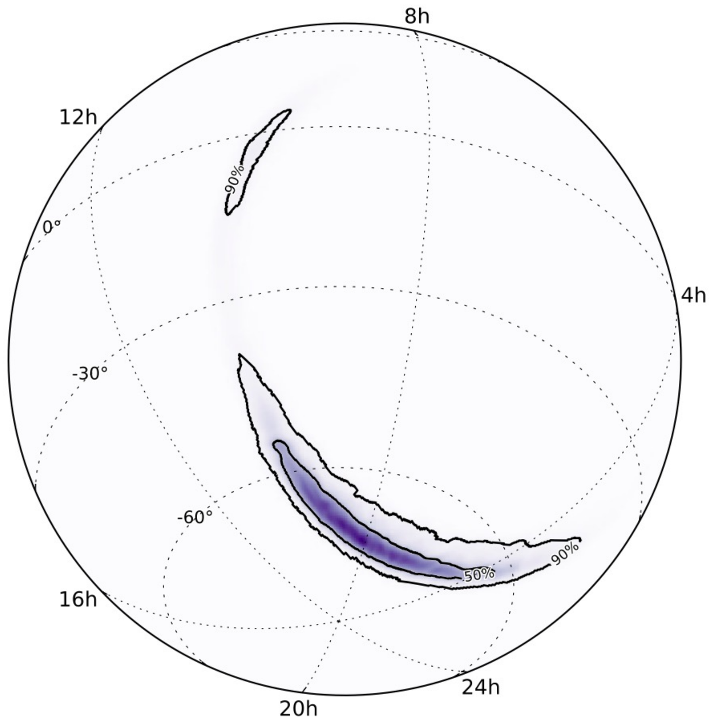
GW signals have the properties of their sources (masses, spins, sky position, etc.)



masses for GW190521



spins for GW190814

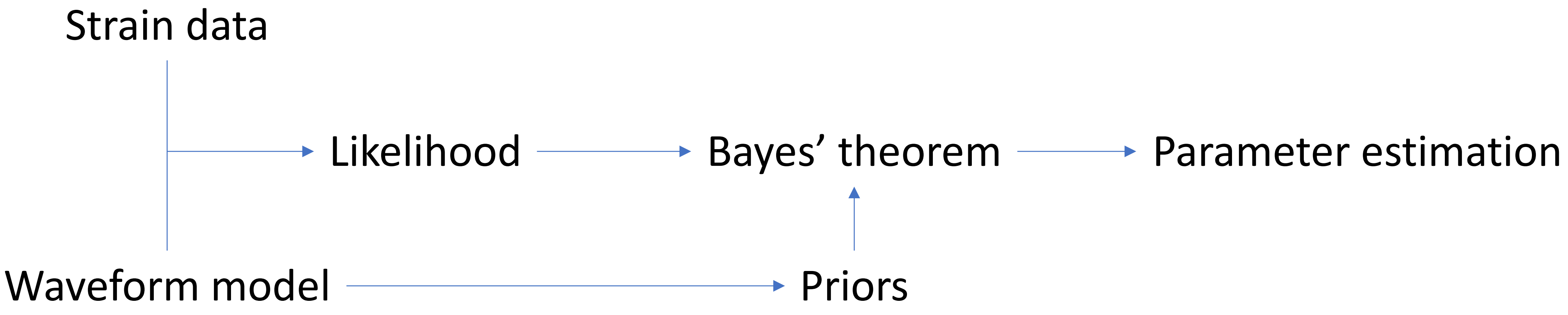


Sky position for GW150914

The goal of parameter estimation is to determine these properties.

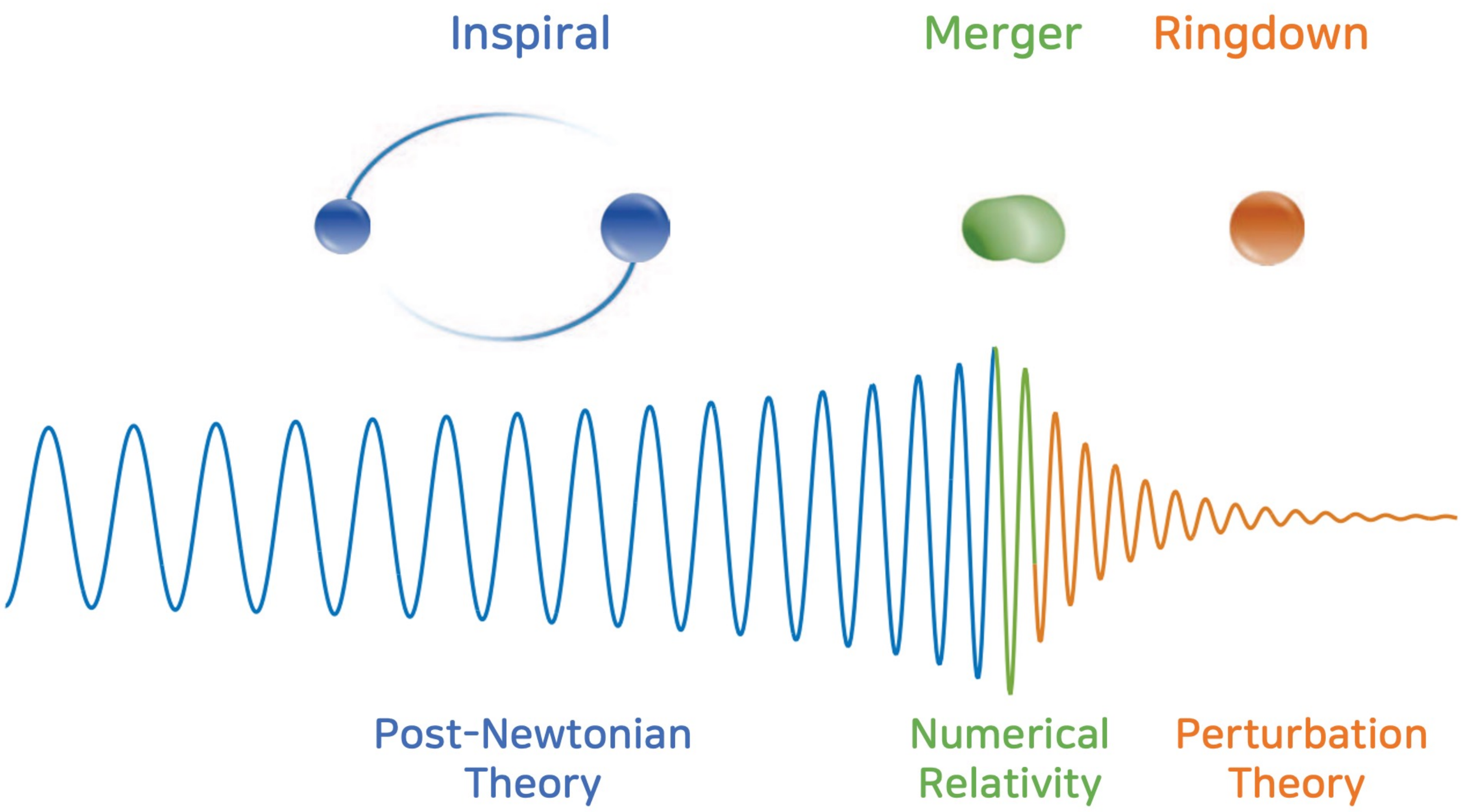
Gravitational Wave

How to estimate parameters from the data?
Use the framework of Bayesian inference.



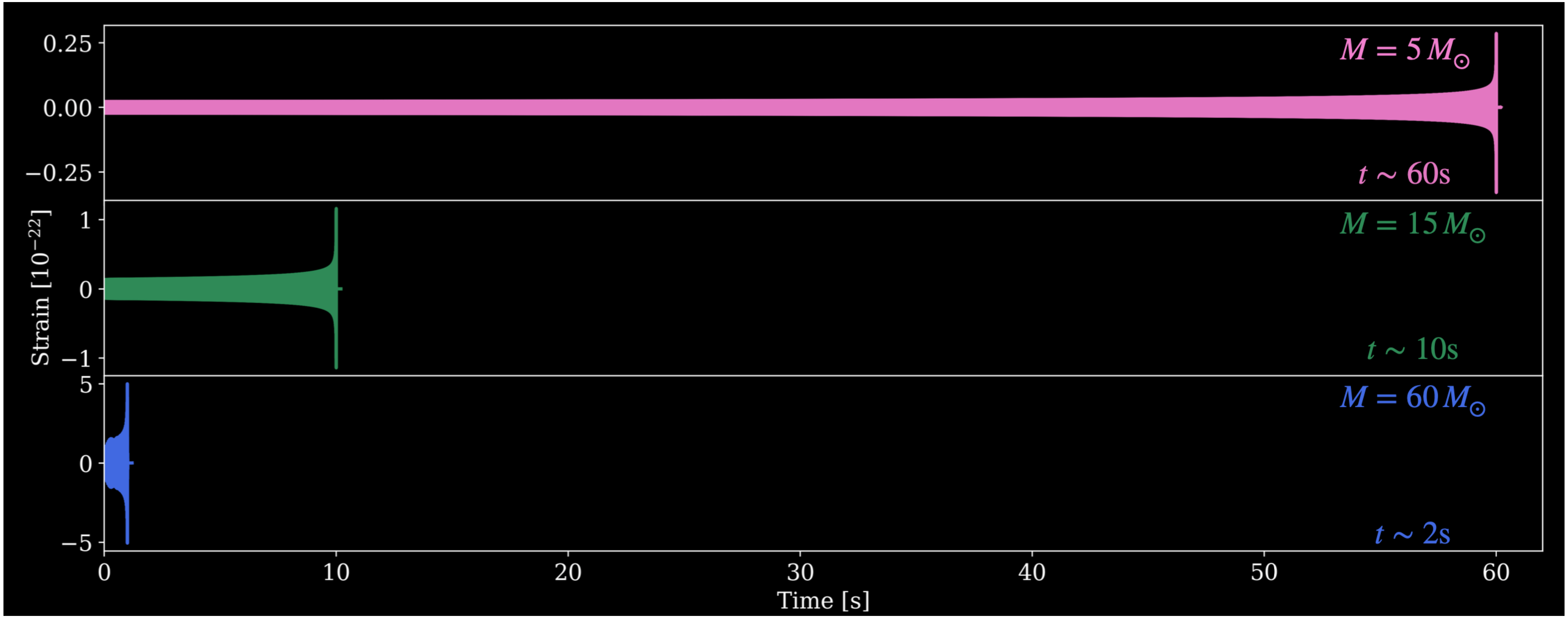
Gravitational Wave : Waveform

Basic CBCs waveform consist of an Inspiral, followed by merger, and then ringdown.



Gravitational Wave

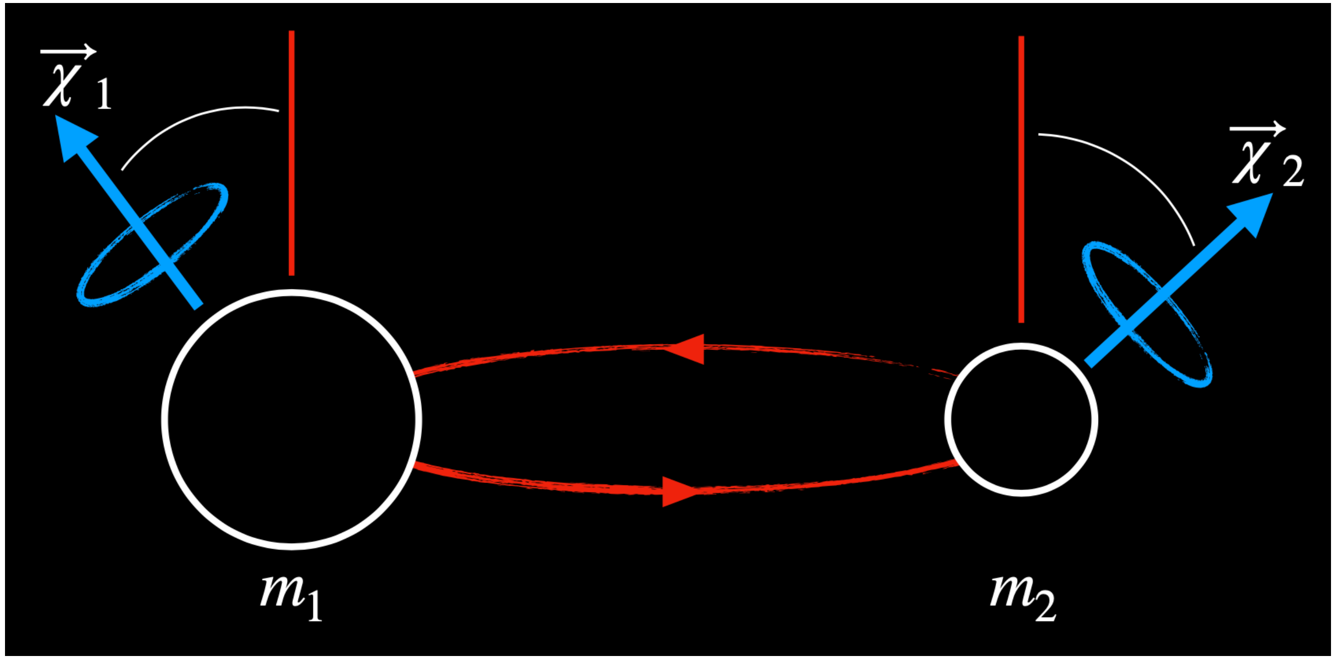
Time spent in-band depends largely on binary mass- higher mass = shorter duration.



Gravitational Wave : intrinsic parameters

8 intrinsic parameters (generally) : 2 masses, 6 spin elements.

For BNS, also have two deformability parameters.



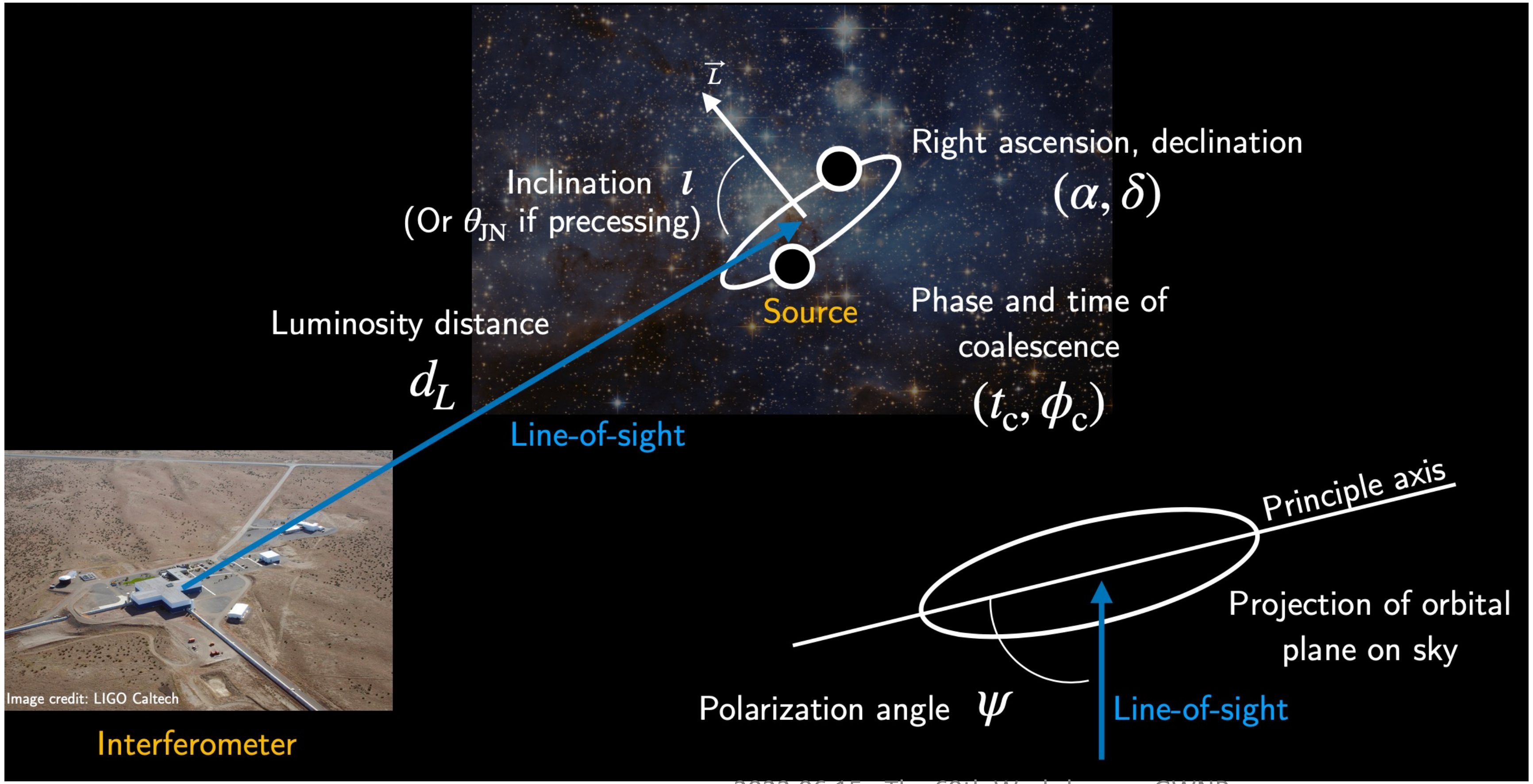
Chirp mass : $M_c = M\eta^{3/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{-1/5}} = \frac{m_1 (q)^{3/5}}{(1+q)^{1/5}}$

Mass ratio : $q = \frac{m_2}{m_1} \leq 1$

Symmetric mass ratio : $\eta = \frac{m_1 m_2}{(m_1+m_2)^2}$

Gravitational Wave : extrinsic parameters

7 extrinsic parameters.



Gravitational Wave : parameters

We have a parameterized model for the signal.
How do we determine these parameters from the data?

intrinsic

Masses : M_c, q

Spins : $\vec{\chi}_1, \vec{\chi}_2$

Tidal : Λ_1, Λ_2

extrinsic

Luminosity distance : d_L

Inclination : θ_{JN}

Sky position : (α, δ)

Polarization angle : Ψ

Coalescence time : t_c

Coalescence phase : ϕ_c

PE : Bayes' theorem

Statement about conditional probabilities.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

PE : Bayes' theorem

For GW parameter estimation, can write Bayes' theorem like this

Posterior

Likelihood

Prior

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

Evidence

Model parameters

Calibrated strain data

Waveform model

PE : Likelihood

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

The probability of observing the data *d* given a signal model *h* with parameters θ.
Approximate the noise as both stationary and Gaussian.

Fourier transforms of data, waveform

$$\ln L(d|\theta,h) \propto -\sum_k \frac{2\left|\tilde{d}_k - \tilde{h}_k(\theta)\right|^2}{S_n(f_k)T}$$

Data segment duration

Sum over frequency bins f_k

Noise power spectral density

PE : prior

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

Represent our assumptions about the model parameters a prior

- Uniform distribution in the masses, spin magnitudes
- Isotopic distribution in spin angles, sky position, inclination
- Distance prior uniform in comoving volume
- Uniform distribution in time, phase, polarization angle

$$\int dm_1 dm_2 = 1$$

$$\mathrm{d}m_1 = \frac{m_1}{M_{ch}} dM_{ch}$$

$$\int dm_1 dm_2 = \int m_1\, dm_1 dq = \int \frac{m_1^2}{M_{ch}} dM_{ch} dq = 1$$

$$\mathcal{P}(M_{ch}, q) = \frac{m_1^2}{M_{ch}} = M_{ch}(1+q)^{2/5} q^{-6/5}$$

$$\int D^2 dD d\theta d\phi = 1$$

$$\mathcal{P}(\mathrm{D}) = D^2$$

$$\mathrm{x} = \log D$$

$$\mathcal{P}(\mathrm{x}) = D^3$$

PE : Evidence

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

Normalizing factor for the posterior distribution.

$$Z(d|h) = \int L(d|\theta,h)\pi(\theta|h)\,d\theta$$

Construct Bayes' factors to compare evidence for one model vs. another

$$\mathcal{B}_Y^X = \frac{Z(d|h_X)}{Z(d|h_Y)}$$

B_{XY}	$2\log B_{XY}$	Evidence for model X
< 1	< 0	Negative (supports model Y)
1 to 3	0 to 2	Not worth more than a bare mention
3 to 12	2 to 5	Positive
12 to 150	5 to 10	Strong
> 150	> 10	Very Strong

PE : posterior

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

Goal : find this posterior distribution for the model parameters
We can evaluate the likelihood over a grid

15 parameters → 10^{15} points

It takes ~ 10^4 yesrs !

PE : posterior

$$P(\theta|d,h) = \frac{L(d|\theta,h)\pi(\theta|h)}{Z(d|h)}$$

15 parameters → 10^{15} points

So, we use a stochastic sampler to infer the posterior distributions.

PE : method (MCMC)

The Markov Chain Monte Carlo (MCMC) method is a computational technique used to estimate complex statistical models or perform numerical integration when traditional methods are not feasible or inefficient.

The Markov chain : Irreducible → Any state could be arrived from any state

Aperiodic → System does not have deterministic cycle

Ergodic → Any Markov Process converge to a unique statistical equilibrium from any state

A sequence of random states for which the probability of a state depends only on the previous state

PE : method (MCMC)

Define the Probability Distribution

Choose a Markov Chain

Initialization

Iteration : Metro-Hasting Algorithm

Burn-in Period

Sampling

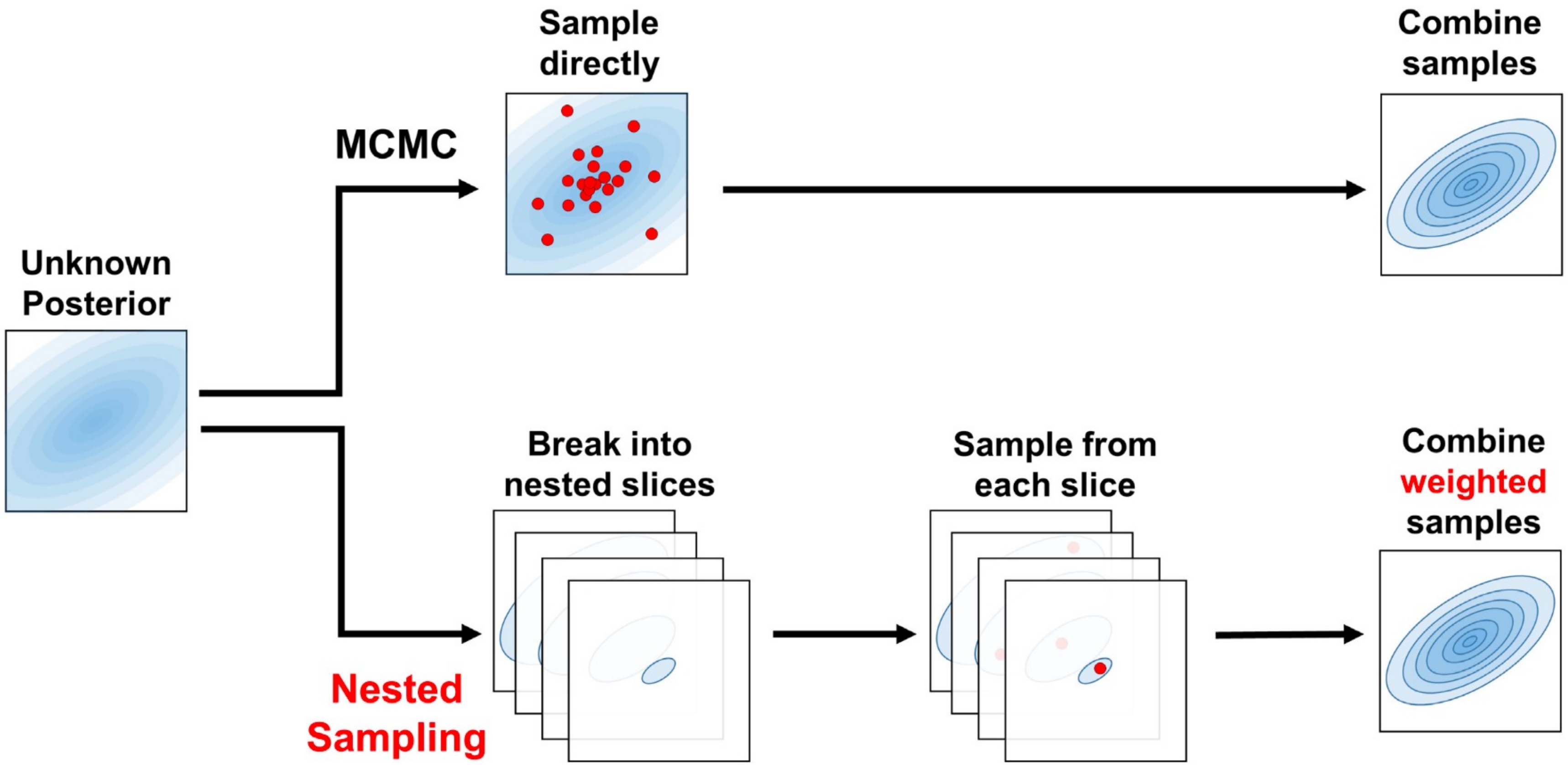
Convergence Assessment

1. Select a starting point $\vec{\theta}_0$ according to the probability distribution $P_0(\vec{\theta})$ and calculate the posterior $\mathrm{P}(\vec{\theta}_0|\vec{d})$.
2. Generate a new parameter set $\vec{\theta}_*$ by using the proposal probability distribution $\mathcal{Q}(\vec{\theta}_*|\vec{\theta}_{t-1})$. Note that the candidate point depends only on $\vec{\theta}_{t-1}$.
3. Calculate the Posterior $\mathrm{P}(\vec{\theta}_*|\vec{d})$.
4. Calculate the acceptance probability for the posterior of the previous (Step1) and the current posterior (Step3). The acceptance probability r_a is as follows:

$$r_a = \min\left[1, \frac{\mathrm{P}(\vec{\theta}_*|\vec{d})\mathcal{Q}(\vec{\theta}_*|\vec{\theta}_{t-1})}{\mathrm{P}(\vec{\theta}_{t-1}|\vec{d})\mathcal{Q}(\vec{\theta}_{t-1}|\vec{\theta}_*)}\right].$$

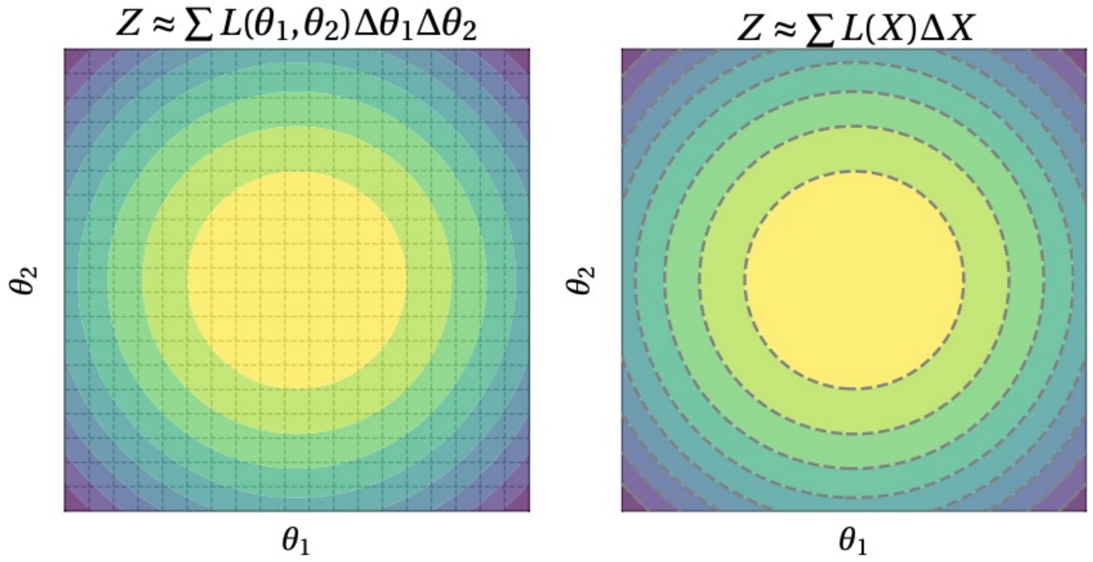
5. Generate a random number between [0,1] and compare it with r_a calculated in Step4. A new candidate is accepted only when the generated random number is less than r_a, then one chooses $\vec{\theta}_t = \vec{\theta}_*$. If the value is greater than r_a, the posterior selection is repeated.

PE : method (Dynesty)

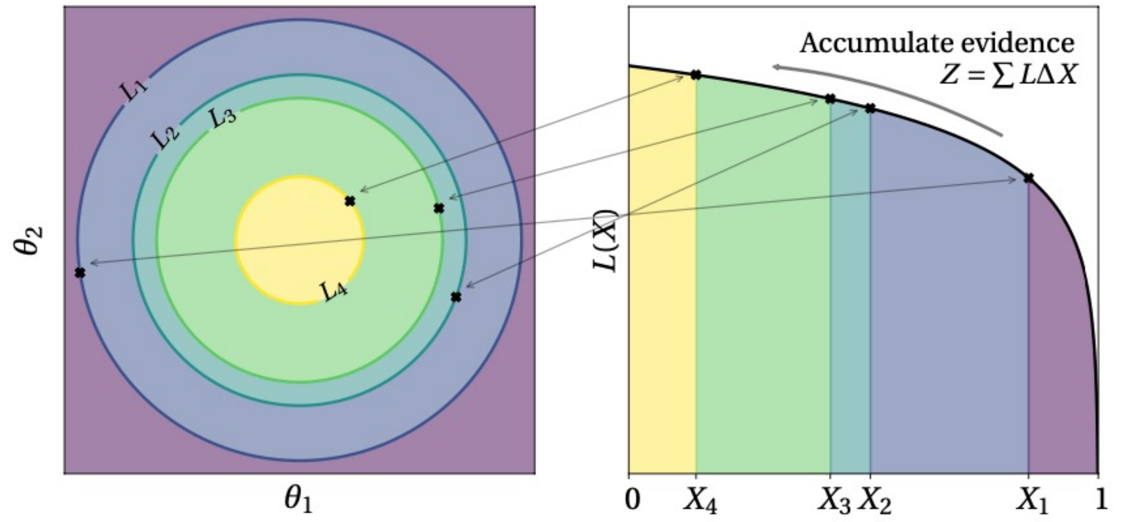


(i) ‘slicing’ the posterior into many simpler distributions,
(ii) sampling from each of those in turn, and
(iii) re-combining the results afterwards.

PE : method (Dynesty)



a | **The NS evidence identity.** The colours represent contours of a two-dimensional likelihood function. Rather than summing over little cubes (left), we combine cubes of similar likelihood together and sum over them (right).



b | **NS on a two dimensional problem.** We show the dead points and their iso-likelihood contours (left) and the corresponding contributions to the evidence integral (right). The volumes X_i are estimated statistically in NS.

Nature Reviews Methods Primers volume 2, Article number: 39 (2022)

PE : method (Dynesty)

Define the Model : Likelihood function

Likelihood Weighted Sampling

Select the Lowest Likelihood Point : DEAD POINT

Prior Mass

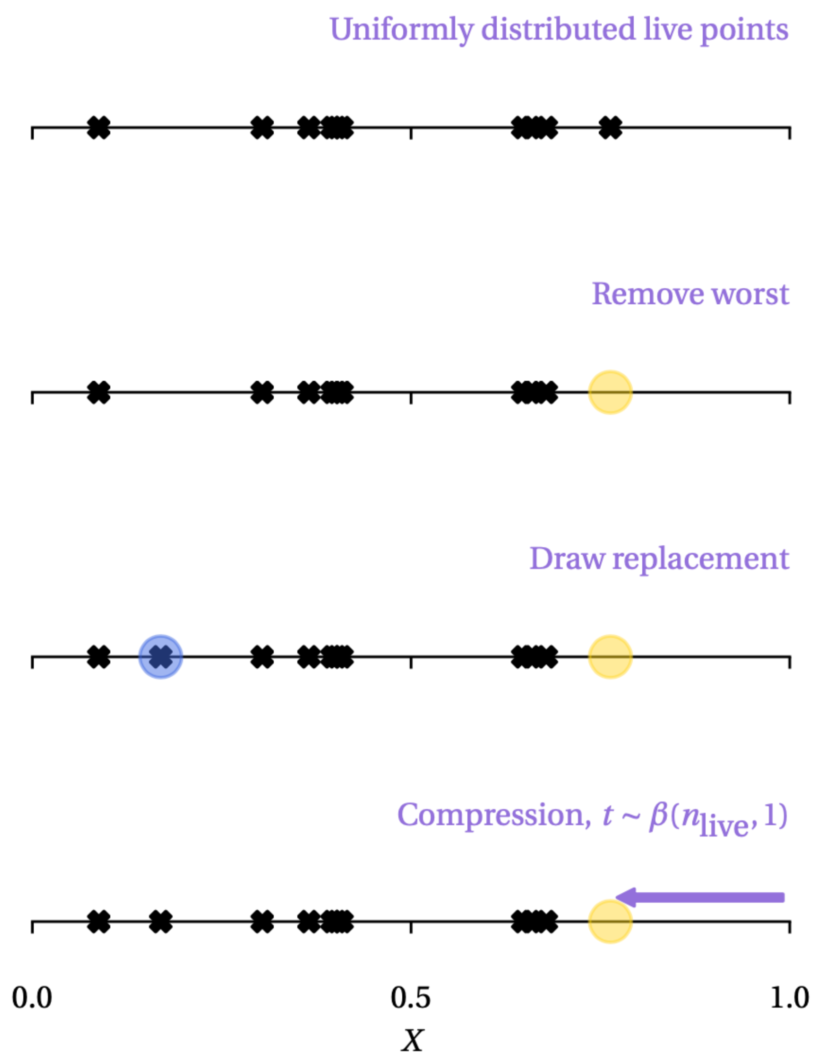
Sampling New Points

Replace the Lowest Likelihood Point

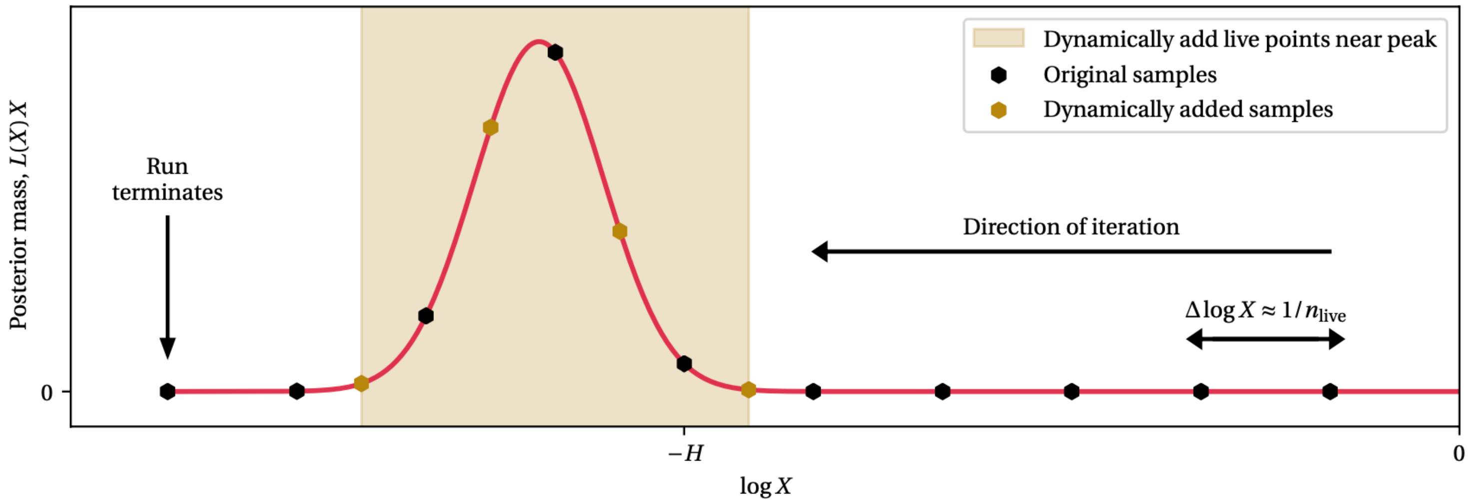
Iterate Until Convergence

Estimate the Evidence

Parameter Exploration



PE : method (Dynesty)

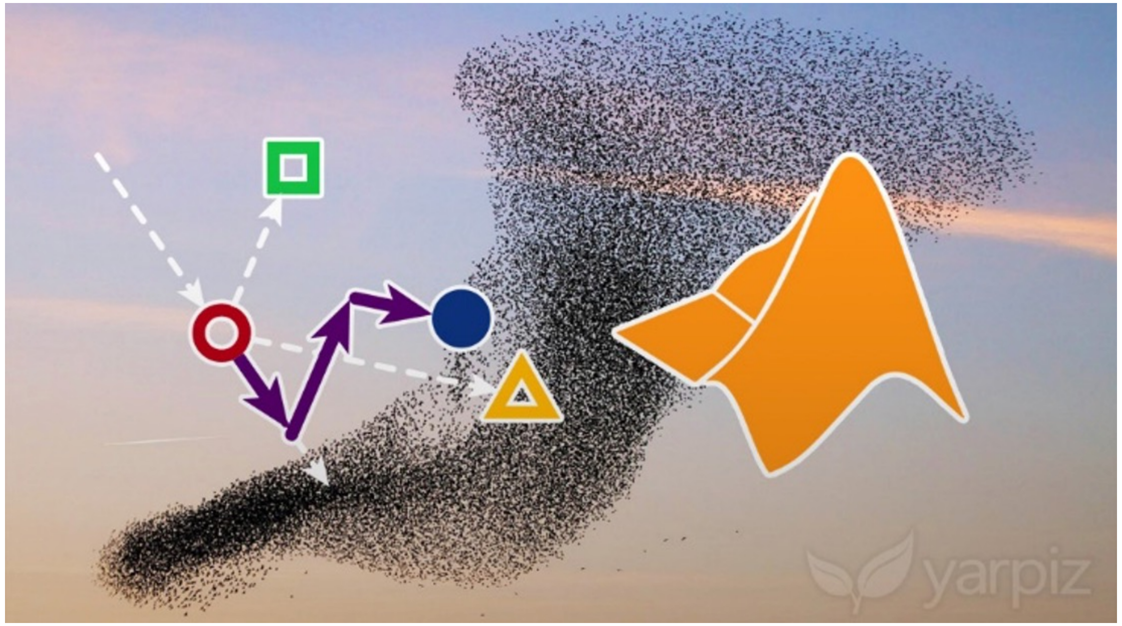


a | **Schematic representation of an NS run.** The curve $L(X)X$ shows the relative posterior mass, the bulk of which lies in a tiny fraction
the volume. Most of the original samples lie in regions with negligible posterior mass. In dynamic NS, we add samples near the peak.

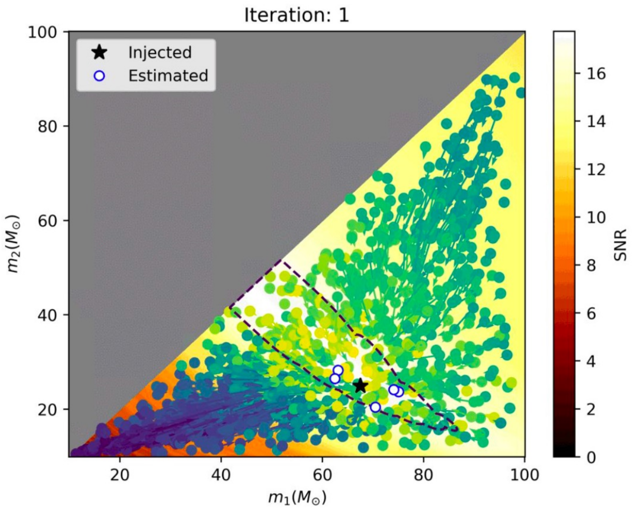
Algorithm 2: Dynamic Nested Sampling

`// Initialize first set of live points.`
Draw K “live” points$\{\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_K\}$ from the prior $\pi(\boldsymbol{\Theta})$.
`// Main sampling loop.`
Set $\mathcal{L}^{\min} = 0$ and $K_0 = K$.
while stopping criterion not met **do**
  `// Get current number of live points.`
  Compute the previous number of live points K and thecurrent number of live points K'.
  if $K' \geq K$ **then**
    `// Add in new live points.`
    while $K' > K$ **do**
      Sample a new point $\boldsymbol{\Theta}'$ from the prior subject to the constraint$\mathcal{L}(\boldsymbol{\Theta}') \geq \mathcal{L}^{\min}$.
      Add $\boldsymbol{\Theta}'$ to the set of live points.
      Set $K = K + 1$.
    end
    `// Replace worst live point.`
    Compute the minimum likelihood $\mathcal{L}^{\min}$ among thecurrent set of K live points.
    Add the kth live point $\boldsymbol{\Theta}_k$associated with $\mathcal{L}^{\min}$to a list of “dead” points.
    Replace $\boldsymbol{\Theta}_k$ with $\boldsymbol{\Theta}'$.
  else
    `// Iteratively remove live points.`
    while $K' < K$ **do**
      Compute the minimum likelihood $\mathcal{L}^{\min}$ among thecurrent set of $K = K'$ live points.
      Add the kth live point $\boldsymbol{\Theta}_k$ associated with $\mathcal{L}^{\min}$to a list of “dead” points.
      Remove $\boldsymbol{\Theta}_k$ from the set of live points.
      Set $K = K - 1$.
    end
  end
  `// Check whether to stop.`
  Evaluate stopping criterion.
end
`// Add final live points.`
while there are live points remaining **do**
  Compute the minimum likelihood $\mathcal{L}^{\min}$ among thecurrent set of live points.
  Add the kth live point $\boldsymbol{\Theta}_k$ associated with $\mathcal{L}^{\min}$to a list of “dead” points.
  Remove $\boldsymbol{\Theta}_k$ from the set of live points.
end

PE : method (PSO)



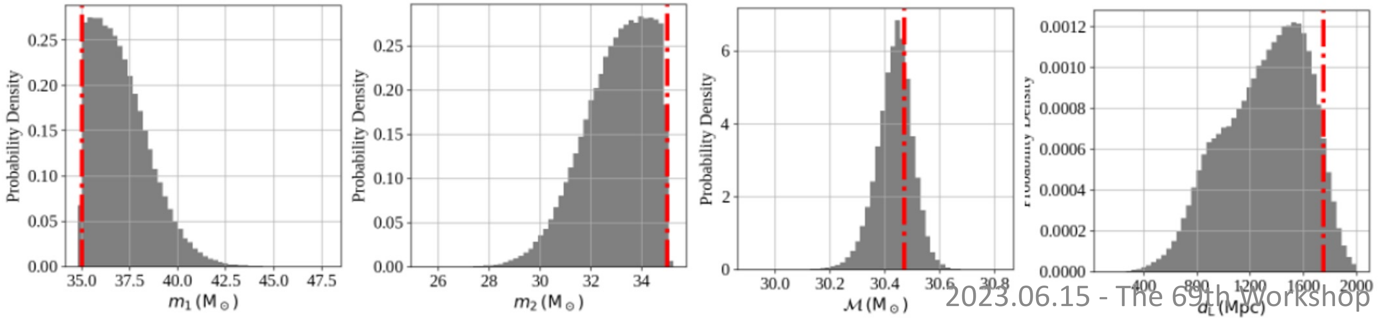
Swarm intelligence for CBCs



- No pre-computed template banks.
- Template points are computed on the fly by improving the SNR / newSNR.
- Explores arbitrary dimensional search parameter space in relatively low cost.
- Exploit it to include eccentricity, spin-precession..

$$\boxed{x_i^d(t+1) = x_i^d(t) + v_i^d(t+1)}$$

$$\boxed{v_i^d(t+1) = \alpha\, r_\alpha v_i^d(t) + \beta\, r_\beta [x_i^d(t) - p_i^d(t)] + \gamma\, r_\gamma [x_i^d(t) - g^d(t)]}$$



PE : method (flowMC)

flowMC: Normalizing-flow enhanced sampling package for probabilistic inference in Jax

Kaze W. K. Wong[1], **Marylou Gabrié**[2, 3], **and Daniel Foreman-Mackey**[1]

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Summary

Across scientific fields, more and more flexible models are required to understand increasingly complex physical processes. However the estimation of models'parameters becomes more challenging as the dimension of the parameter space grows. A common strategy to explore parameter space is to sample through a Markov Chain Monte Carlo (MCMC). Yet even MCMC methods can struggle to faithfully represent the parameter space when only relying on local updates.

Eccentric GW waveform with aligned-spin

An accurate waveform model (eccentricity and spin)
to measure the physical quantity of a gravitational wave source

→ All of the GWs waveforms (amplitude, phase).
→ Distance measurement accuracy.

$$f_{\mathrm{GW}} = 2 f_{\mathrm{orb}}$$

$$f_{\mathrm{GW}} = \frac{1}{\pi}\sqrt{\frac{GM}{a^3}}$$

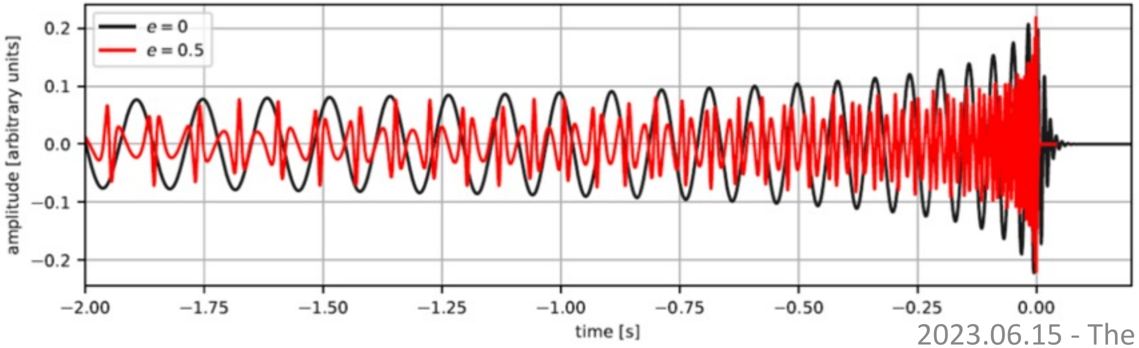
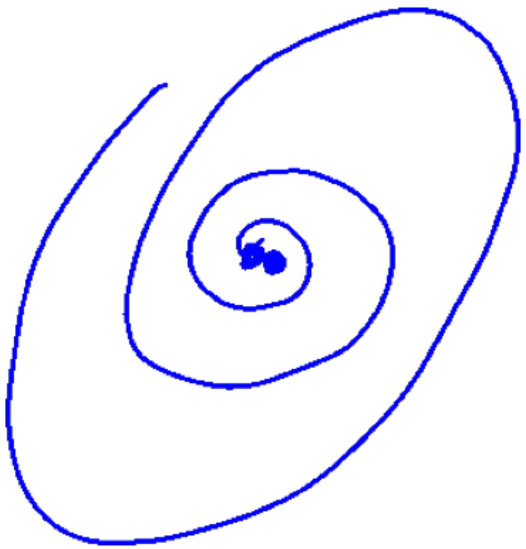


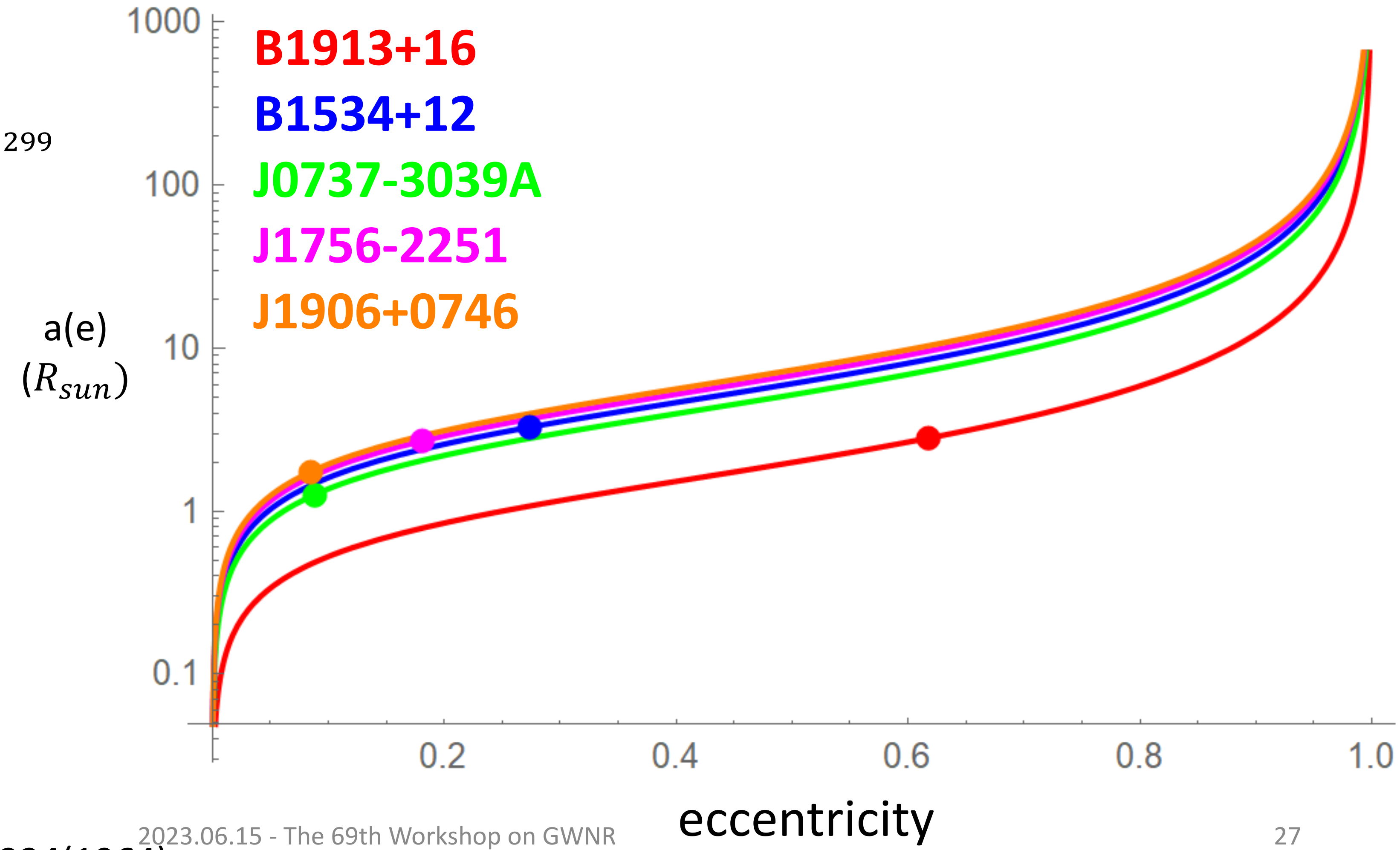
Figure 1. Examples of gravitational waveforms for a $10\,M_\odot - 10\,M_\odot$ BBH system with eccentricities 0 (black) and 0.5 (red).



$$\frac{de_t}{dt} = -\frac{\eta}{15M}\left(\frac{M}{r_p}\right)^4 \frac{e_t(1-e_t^2)^{3/2}}{(1+e_t)^4}\left(304 + 121 e_t^2\right)$$

Orbital evolution of known NS-NS merger in our Galaxy (following Peters 1964)

$$a(e) = \frac{c_0 e^{12/19}}{(1-e^2)}\left[1+\frac{121}{304}e^2\right]^{870/2299}$$



eq 5.11 - Peters's paper Phys. Rev. 136,B1224(1964)

Eccentricity evolution

P_{orb} : orbital period (in days)

$f_{gw,i} = \frac{2}{P_{ord}}$: current fundamental gravitational-wave frequency (mHz)

e_i : current eccentricity

e_t (5mHz) : eccentricity at eLISA band, $e_t(10Hz)$: eccentricity at LIGO band

$$\frac{f_{gw}}{f_{gw,i}} = \left(\frac{e_i}{e_t}\right)^{18/19} \left(\frac{1-e_t^2}{1-e_i^2}\right)^{3/2} \left(\frac{304+121e_i^2}{304+121e_t^2}\right)^{1305/2299}$$

GW band

Source	P_{orb} (days)	$f_{\text{gw,i}}$ (mHz)	e_i	e_t (5 mHz)	e_t (10 Hz)
J0737 − 3039	0.102 251 562 48	0.226	0.087 777 5	0.003 39	1.11×10^{-6}
J1906 + 0746*	0.165 993 046 83	0.139	0.085 302 8	0.001 98	6.48×10^{-7}
J1756 − 2251	0.319 633 901 43	0.0724	0.180 594	0.002 20	7.20×10^{-7}
B1913 + 16	0.322 997 448 911	0.0717	0.617 133 4	0.0162	5.32×10^{-6}
B2127 + 11C	0.335 282 048 28	0.0690	0.681 395	0.0220	7.23×10^{-6}
B1534 + 12	0.420 737 298 879	0.0550	0.273 677 52	0.002 70	8.85×10^{-7}

Most values for orbital period and current eccentricity are taken from the Australia Telescope National Facility (ATNF) pulsar catalog.

Table2 and eq1.2 in Moore et al, PhysRevD.93.124061(2016)

Eccentric inspiral GW waveform : TaylorF2Ecc

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\Psi(f)}$$

$$\mathcal{A} = -M\sqrt{\frac{5\pi}{96}}\left(\frac{M}{D}\right)\sqrt{\eta}(\pi M f)^{-7/6}\times[(1+C^2)^2 F_+^2 + 4C^2 F_\times^2]^{1/2}$$

$$\Psi(f) = \phi_c + 2\pi f t_c + \frac{3}{128\eta v^5}\left(1 + \Delta\Psi_{3.5PN}^{circ.} + \Delta\Psi_{4PN}^{spin,circ.} + \Delta\Psi_{3PN}^{ecc.}\right)$$

$$\Delta\Psi_{4PN}^{spin,circ.} = 4\beta_{1.5}v^5 - 10\sigma v^4 + v^5 \ln v^3\left[\frac{40}{9}\beta_{2.5} - \beta_{1.5}\left(\frac{3715}{189} + \frac{220}{9}\eta\right)\right] + \mathcal{P}_6 v^6 + \mathcal{P}_7 v^7 + \mathcal{P}_8 v^8$$

$$\Delta\Psi_{3PN}^{ecc.} = -\frac{2355}{1462} e_0^2 \left(\frac{v_0}{v}\right)^{19/3}\left[1 + v^2\left(\frac{299076223}{81976608} + \frac{18766963}{2927736}\eta\right) + v_0^2\left(\frac{2833}{1088} - \frac{197}{36}\eta\right) + \cdots + \mathcal{O}(v^6)\right]$$

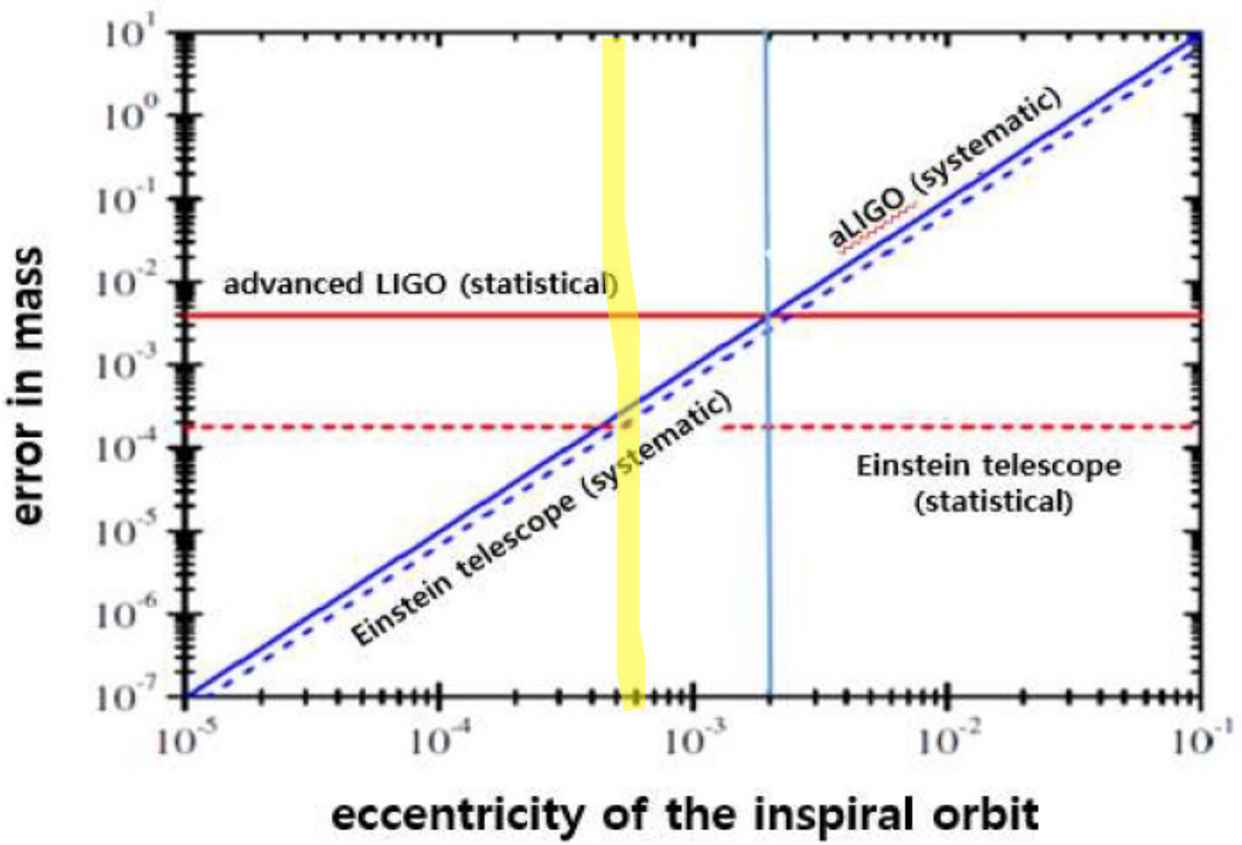
Parameter estimation with TaylorF2Ecc

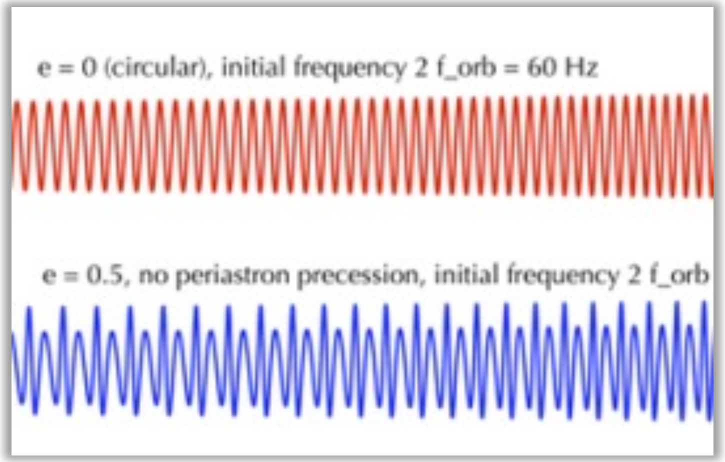
- GW waveform model : TaylorF2Ecc (GW signal as well as template ➔ systematics)
- Detector : LIGO Livingston (L1), LIGO Hanford (H1), Virgo (V1)
- Noise model: no noise and Gaussian noises are considered.
- frequency range : fGW = [25, ISCO frequency] Hz for different **inspirals**

Source	M_c (M_{sun})	η	dist. (Mpc)	SNR	ι (rad)	polari-zation ψ (rad)	orb. phase ϕ_c (rad)	RA (rad)	dec (rad)
GW151226	9.66653	0.230348	800	21	$\frac{\pi}{4}$	2.606	3.31	0.645	0.575
GW170608	8.48799	0.241646	730	20					
GW200105	3.593414	0.144976	360	21					

advaced LIGO(10 Hz) ➔ constraining/possible measurement

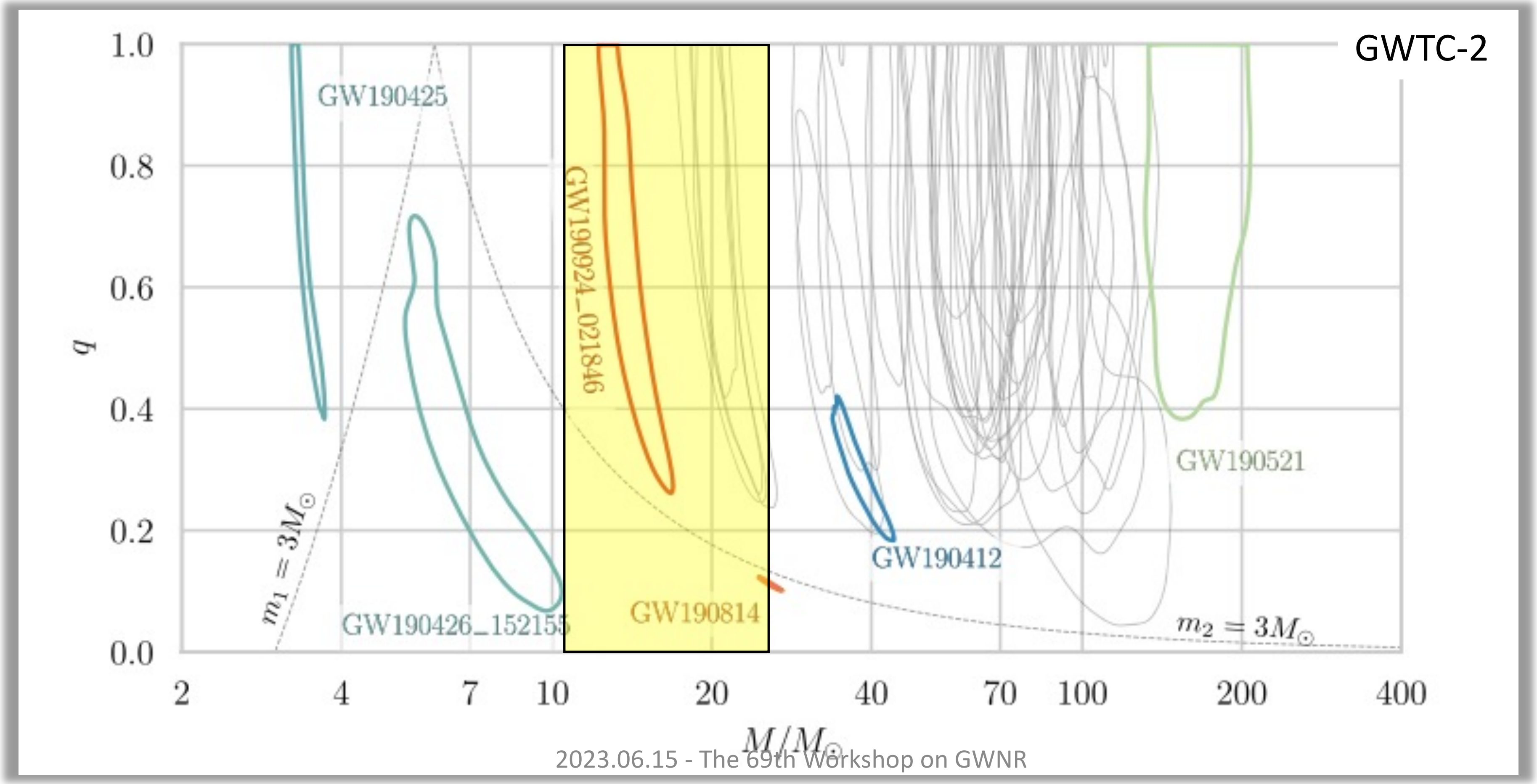
Einstein telescope(1 Hz) ➔ plausible to find eccentric binaries



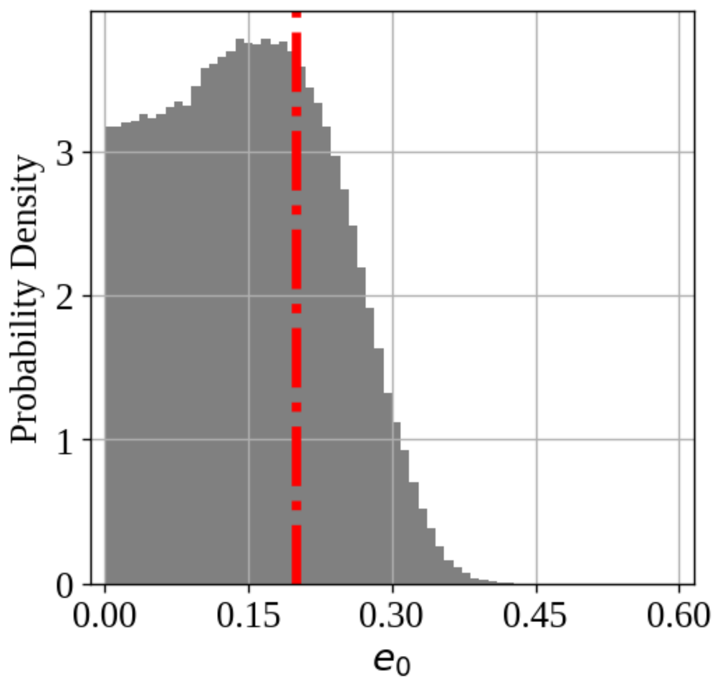


GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run

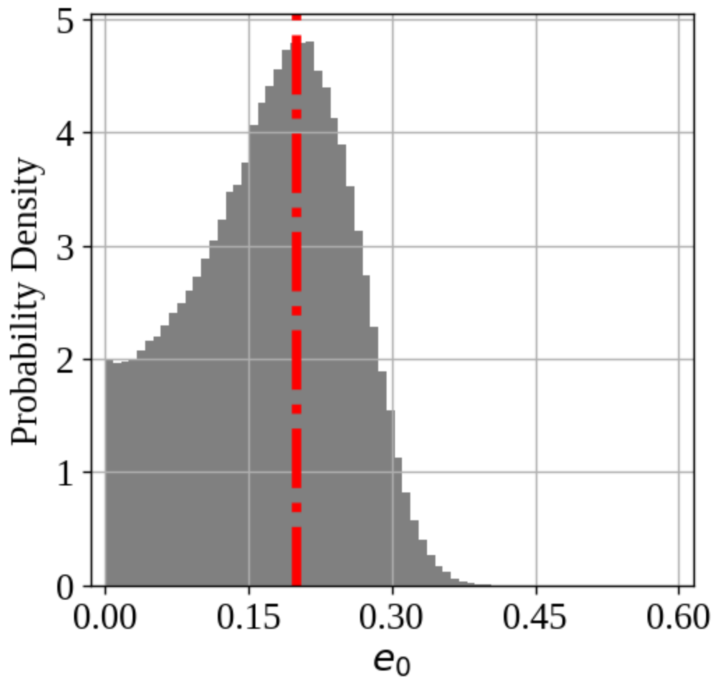
"inspiral-dominant" compact binaries ➔ "longer" signal duration



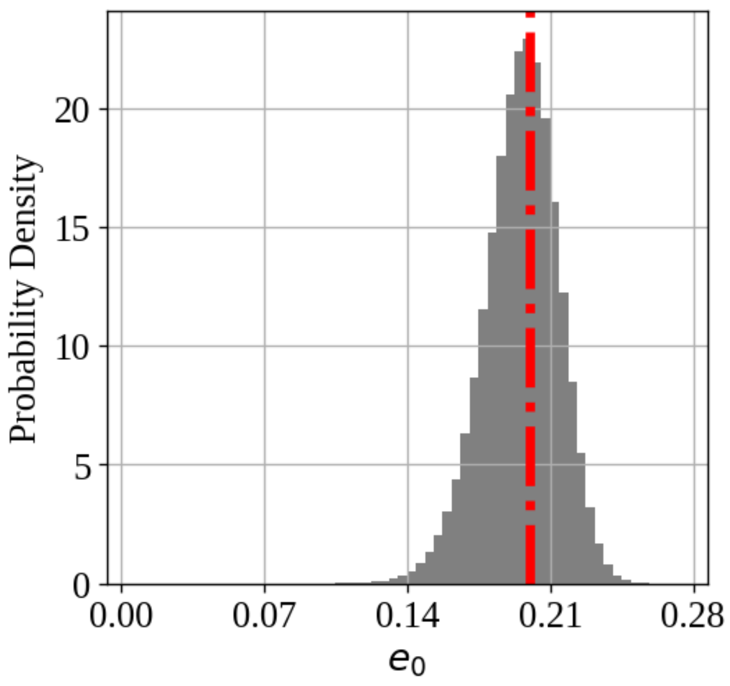
PE Results for inspirals. “lightest BH-BH” or “BH-NS”



GW151226-like system
m_1= $14.9 M_\odot$, m_2= $8.4 M_\odot$
~100cyles



GW170608-like system
m_1= $11.8 M_\odot$, m_2= $8.1 M_\odot$
~ 130cycles



GW200105-like system
m_1= $9.4 M_\odot$, m_2= $2.0 M_\odot$
~ 570cycles

GW151226-like system PE Results

chirp mass ($\mathcal{M}$) :

$$\mathcal{M} = M\eta^{3/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{-1/5}}$$

eta (η) : symmetric mass ratio

$$\eta = \frac{m_1 m_2}{(m_1+m_2)^2}$$

larger values of e(= 0.2) and spin magnitudes are helpful to better determine the chirp mass.

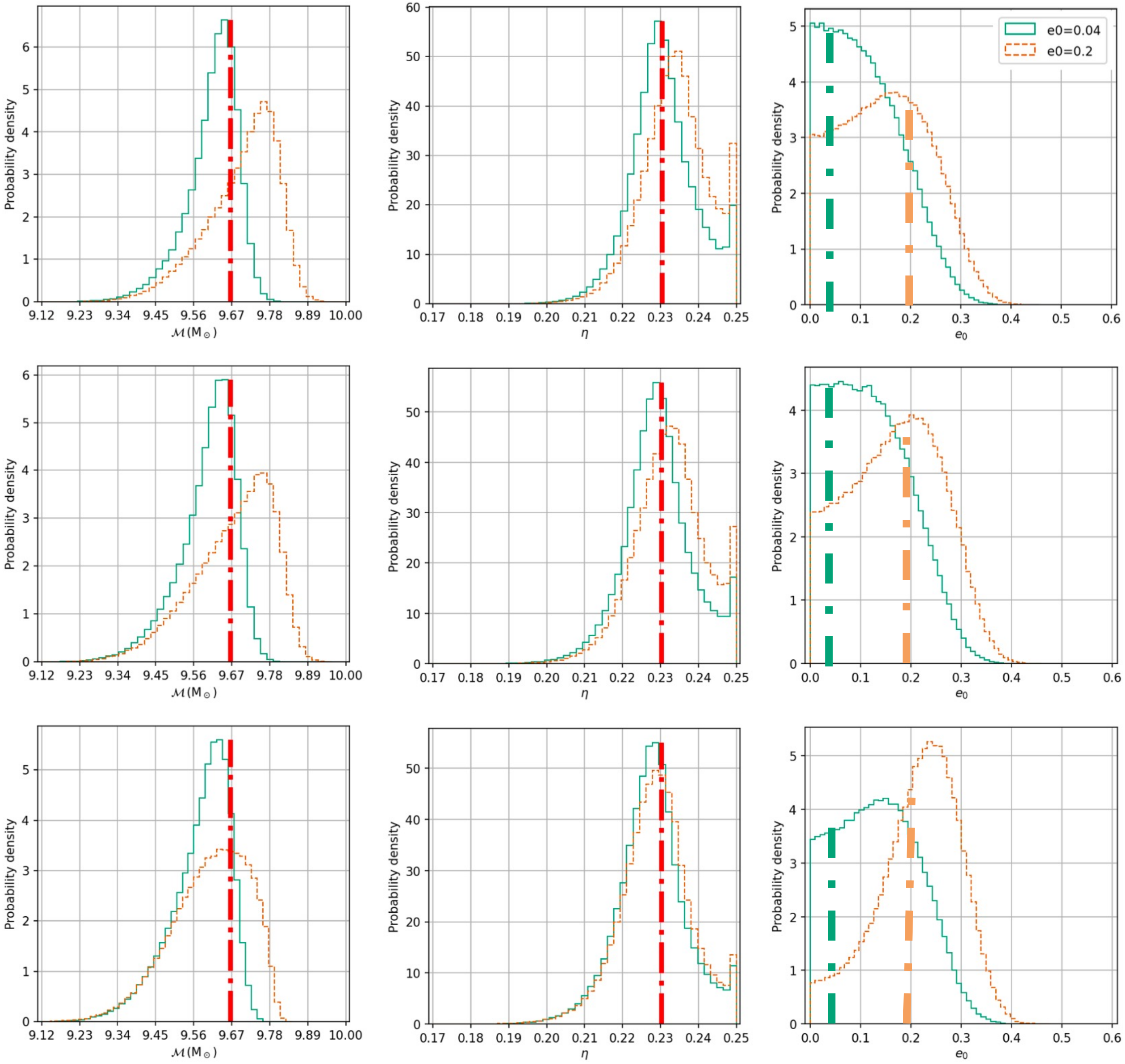
equal-mass confusion can be reduced if spin magnitudes are larger

a1z=a2z

0.3

0.7

0.9



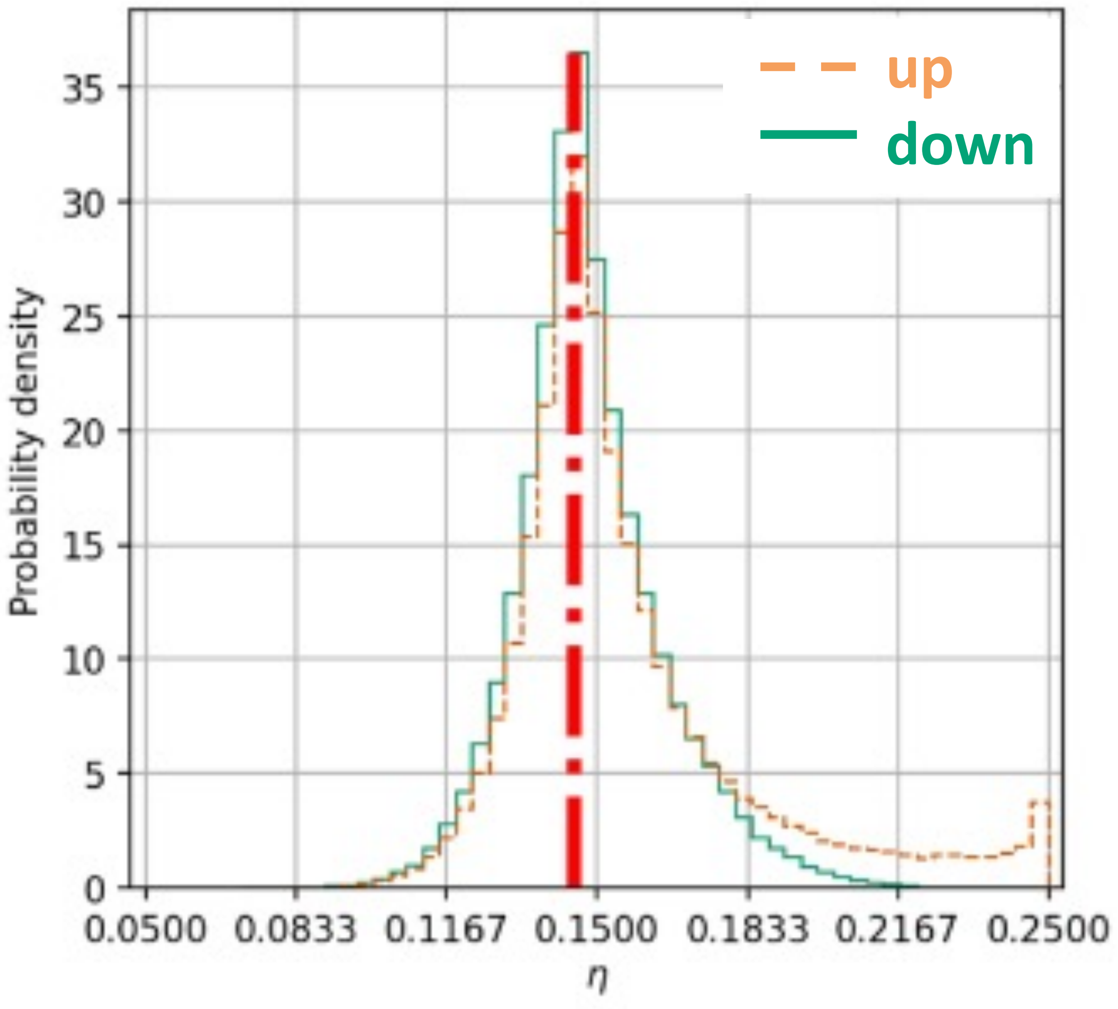
PE Result for the symmetric mass ratio of GW200105 (BH-NS)

Consider the BH spin only: a1z

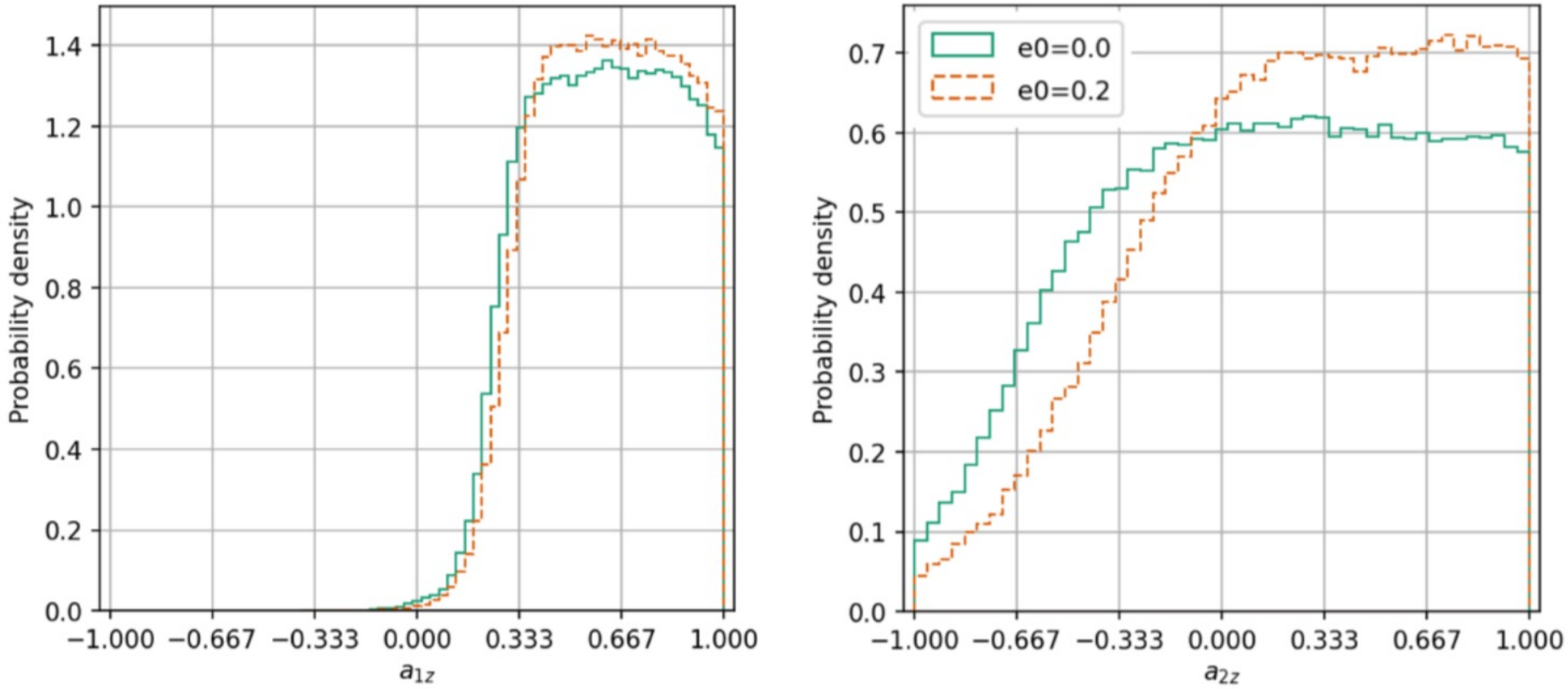
Compare two examples:
a1z = **+0.5 (up)** or **a1z=** **-0.5 (down)**.

When a black hole's spin is **anti-aligned** with its orbital angular momentum, the equal-mass posterior can be completely ruled-out.

BH spin median ~ 0.6 (based on GWTC papers)



PE Results for spin of GW151226 (BH-BH)



The accuracy of a1z is not sensitive to the size of the eccentricity.

The a2z is slightly better constrained if the orbit is more eccentric.

Summary

- MCMC PE for GW inspirals with TaylorF2Ecc, design PSDs

➔ study systematics

- Phase corrections due to aligned spin(s) and e0

- Existence of aligned spin(s) are helpful to better constrain e0 and mass parameters.