1+3 formalism

Chan Park (SNU)

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Overview

- Part 1: General Relativity with a Linear Connection
- Part 2: 1+3 Formalism Development

Part 1: General Relativity with a Linear Connection

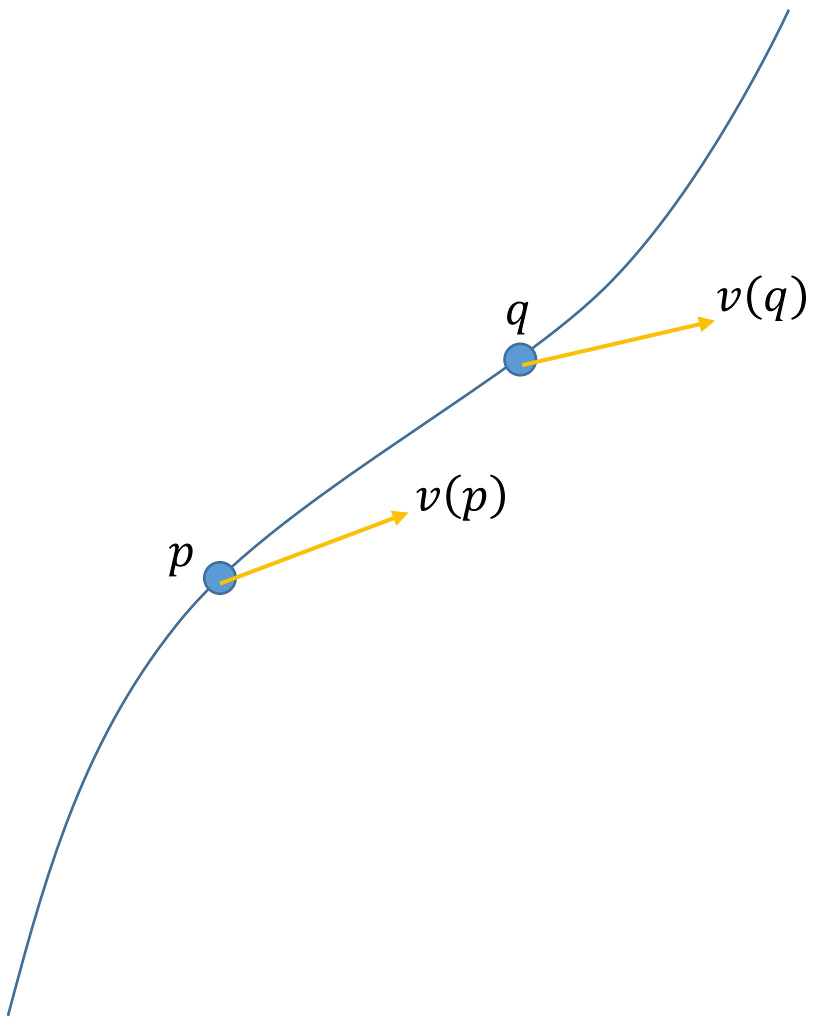
Linear Connection,
Torsion,
and Riemann Curvature

Abstract Index Notation

- Abstract indices: $a, b, c, \cdots$
- Contraction
  - $v^a \omega_a = \boldsymbol{v}(\boldsymbol{\omega}) = \boldsymbol{\omega}(\boldsymbol{v}) = v^\mu \omega_\mu$
  - $T^{a\cdots}{}_{b\dots} \omega_a \cdots v^b \cdots = \boldsymbol{T}(\boldsymbol{\omega}, \cdots, \boldsymbol{v}, \cdots) = T^{\mu\cdots}{}_{\nu\dots} \omega_\mu \cdots v^\nu \cdots$
- Coordinate Basis
  - $\delta^\alpha{}_\beta = (dx^\alpha)_a \left(\partial/\partial x^\beta\right)^a$ $\qquad \delta^a{}_b = (\partial/\partial x^\mu)^a (dx^\mu)_b$
  - $v^a = v^\mu (\partial/\partial x^\mu)^a$ $\qquad \omega_a = \omega_\mu (dx^\mu)_a$
  - $T^{a\cdots}{}_{b\dots} = T^{\mu\cdots}{}_{\nu\dots} (\partial/\partial x^\mu)^a \cdots (dx^\nu)_b$
- Pros
  - Clear distinction between tensors and components
  - Independence of coordinates and bases
  - Good readability

Differentiation of Vectors

- $\Delta \boldsymbol{v} = \boldsymbol{v}(q) - \boldsymbol{v}(p) = ?$

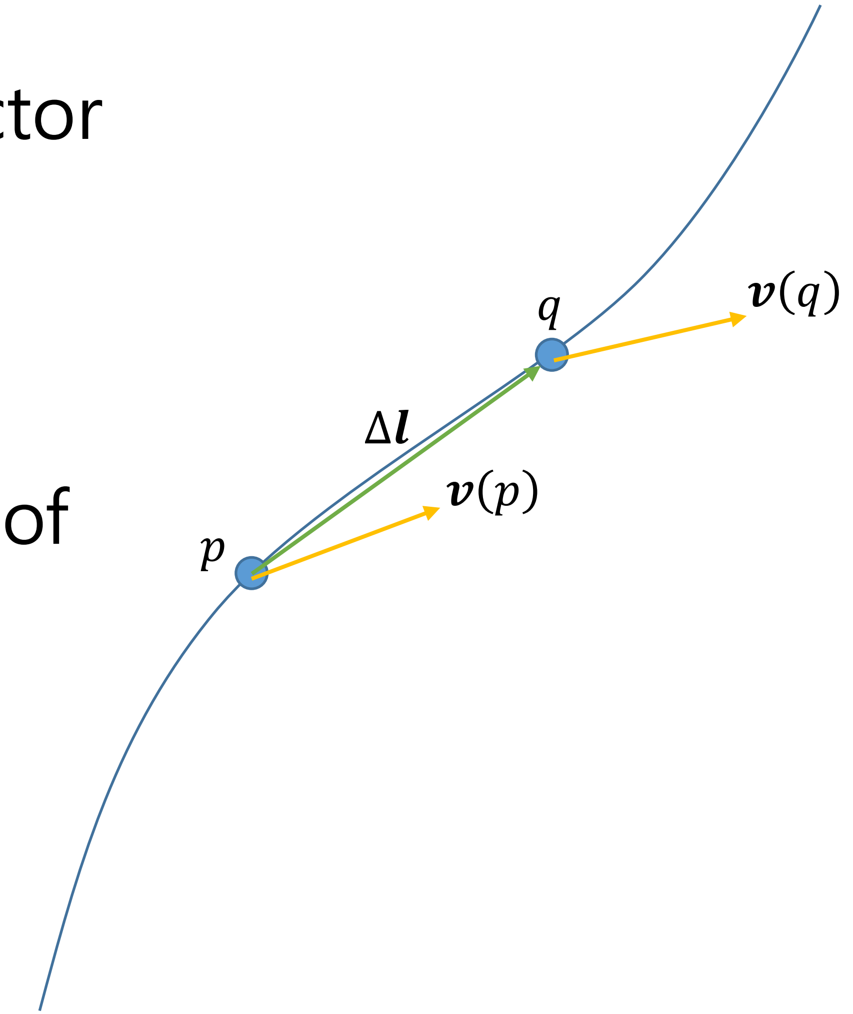


Linear (Affine) Connection

- 1. ∇ : vector field × vector field → vector field
  - $\nabla(\boldsymbol{u},\boldsymbol{v})(\boldsymbol{\omega}) = (\nabla_v \boldsymbol{u})(\boldsymbol{\omega}) = (\nabla \boldsymbol{u})(\boldsymbol{\omega},\boldsymbol{v})$
  - $\omega_a(\nabla_v u)^a = (\nabla u)^a{}_b \omega_a v^b = \omega_a v^b \nabla_b u^a$
- 2. slot 2 (Linearity over scalar field f)
  - $\nabla(\boldsymbol{u},\boldsymbol{v} + f\boldsymbol{w}) = \nabla(\boldsymbol{u},\boldsymbol{v}) + f\nabla(\boldsymbol{u},\boldsymbol{w})$
  - $(v^b + f w^b)\nabla_b u^a = v^b \nabla_b u^a + f w^b \nabla_b u^a$
- 3. slot 1 (Leibniz Rule)
  - $\nabla(\boldsymbol{u} + f\boldsymbol{v},\boldsymbol{w}) = \nabla(\boldsymbol{u},\boldsymbol{w}) + f\nabla(\boldsymbol{v},\boldsymbol{w}) + \boldsymbol{v}\boldsymbol{w}(f)$
  - $w^b \nabla_b (u + f v^a) = w^b \nabla_b w + f w^b \nabla_b v^a + v^a w^b (df)_b$

Parallel Transportation via Connection ∇

- $\Delta\boldsymbol{l}$: infinitesimal displacement vector from p to q
- $\Delta\boldsymbol{v} = \nabla(\boldsymbol{v}, \Delta\boldsymbol{l})$
- $(\Delta v)^a = (\Delta l)^b \nabla_b v^a$
- Freedom to choose $\Delta\boldsymbol{v}$ = Degree of freedom of connection



Extension of ∇ to any Type of Tensors

- 1. Gradient: $v^a \nabla_a f = \boldsymbol{v}(f) = v^a (df)_a$
- 2. Leibniz Rule: $\nabla(\boldsymbol{TU}) = \boldsymbol{T}\nabla\boldsymbol{U} + \boldsymbol{T}\nabla\boldsymbol{U}$
- $v^a \nabla_b \omega_a = \nabla_b(\omega_a v^a) - \omega_a \nabla_b v^a$
- $\omega^1_{a_1} \cdots \omega^k_{a_k} v^{b_1}_1 \cdots v^{b_l}_l \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \nabla_c\left(T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \omega^1_{a_1} \cdots \omega^k_{a_k} v^{b_1}_1 \cdots v^{b_l}_l\right)$

$$- \sum_{i=1}^{k} T^{a_1 \cdots a_i \cdots a_k}{}_{b_1 \cdots b_l}\, \omega^1_{a_1} \cdots \nabla_c \omega^i_{a_i} \cdots \omega^k_{a_k} v^{b_1}_1 \cdots v^{b_l}_l$$

$$- \sum_{i=1}^{l} T^{a_1 \cdots a_k}{}_{b_1 \cdots b_i \cdots b_l} \omega^1_{a_1} \cdots \omega^k_{a_k} v^{b_1}_1 \cdots \nabla_c v^{b_i}_i \cdots v^{b_l}_l$$

Torsion Tensor

- Definition
  - $\boldsymbol{T}(\boldsymbol{u},\boldsymbol{v}) \equiv \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}]$
  - $[T(u,v)]^a \equiv u^b\nabla_b v^a - v^b\nabla_b v^a - [u,v]^a = u^a v^b(\nabla_b\nabla_a - \nabla_a\nabla_b)f$
- Anti Symmetricity
  - $\boldsymbol{T}(\boldsymbol{u},\boldsymbol{v}) = -\boldsymbol{T}(\boldsymbol{v},\boldsymbol{u})$
- Bilinearity over scalar field f
  - $\boldsymbol{T}(f\boldsymbol{u}+\boldsymbol{t},\boldsymbol{v}) = f\boldsymbol{T}(\boldsymbol{u},\boldsymbol{v})+\boldsymbol{T}(\boldsymbol{t},\boldsymbol{v})$
  - $[T(u,v)]^a = T^a{}_{bc}u^b v^c$
- Compact Form
  - $T^a{}_{bc}\nabla_a f = -2\nabla_{[b}\nabla_{c]}f$

Riemann Curvature Tensor

- Definition
  - $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}$
  - $[R(u,v)w]^a \equiv u^c\nabla_c\left(v^d\nabla_d w^a\right) - v^d\nabla_d(u^c\nabla_c w^a) - [u,v]^b\nabla_b w^a$
    $= u^c v^d[(\nabla_c\nabla_d - \nabla_c\nabla_d)w^a - T^b{}_{cd}\nabla_b w^a]$
- Anti Symmetricity
  - $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = -\boldsymbol{R}(\boldsymbol{v},\boldsymbol{u})\boldsymbol{w}$
- Bilinearity over scalar field f
  - $\boldsymbol{R}(f\boldsymbol{u}+\boldsymbol{t},\boldsymbol{v})\boldsymbol{w} = f\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} + \boldsymbol{R}(\boldsymbol{t},\boldsymbol{v})\boldsymbol{w}$
  - $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})(f\boldsymbol{w}+\boldsymbol{t}) = f\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} + \boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{t}$
  - $[R(u,v)w]^a = R^a{}_{bcd}w^b u^c v^d$
- Compact Form
  - $R^a{}_{bcd}w^b = 2\nabla_{[c}\nabla_{d]}w^a + T^b{}_{cd}\nabla_b w^a$

Useful Formula

- Commutation of Connection
  - $(\nabla_c\nabla_d - \nabla_d\nabla_c)S^a{}_b = S^e{}_b R^a{}_{ecd} - S^a{}_e R^e{}_{bcd} - T^e{}_{cd}\nabla_e S^a{}_b$
- Lie Derivatives
  - $(\mathcal{L}_u S)^a{}_b = u^c\nabla_c S^a{}_b + S^c{}_b\left(T^a{}_{cd}u^d - \nabla_c u^a\right) - S^a{}_c\left(T^c{}_{bd}u^d - \nabla_b u^c\right)$
- Exterior Derivatives
  - $(d\omega)_{a_1\cdots a_{p+1}} = (p+1)\left(\nabla_{[a_1}\omega_{a_2\cdots a_{p+1}]} + \frac{1}{2}p\omega_{b[a_1\cdots a_{p-1}}T^b{}_{a_p a_{p+1}]}\right)$
- Bianchi Identities
  - $R^a{}_{[bcd]} = \nabla_{[b}T^a{}_{cd]} + T^a{}_{e[b}T^e{}_{cd]}$
  - $\nabla_{[e}R^a{}_{|b|cd]} = -R^a{}_{bf[e}T^f{}_{cd]}$

Metric Connection

- Definition
  - $\nabla_c g_{ab} = 0$ (# of eqs = 4 x 10 = 40)
- Properties
  - $\nabla_e \epsilon_{abcd} = 0$
    - $0 = \nabla_e(\epsilon^{abcd}\epsilon_{abcd}) = 2\epsilon^{abcd}\nabla_e\epsilon_{abcd}$
  - $R_{(ab)cd} = 0$
    - $0 = 2\nabla_{[c}\nabla_{d]}g_{ab} = -g_{eb}R^e{}_{acd} - g_{ae}R^e{}_{bcd}$

Torsion Free Connection

- Definition
  - $T^{a}{}_{bc} = 0$ (# of eqs = 4 x 6 = 24)
- Commutation of Connection
  - $(\nabla_c \nabla_d - \nabla_d \nabla_c) S^{a}{}_{b} = S^{e}{}_{b} R^{a}{}_{ecd} - S^{a}{}_{e} R^{e}{}_{bcd}$
- Lie Derivatives
  - $(\mathcal{L}_u S)^{a}{}_{b} = u^c \nabla_c S^{a}{}_{b} - S^{c}{}_{b} \nabla_c u^a + S^{a}{}_{c} \nabla_b u^c$
- Exterior Derivatives
  - $(d\omega)_{a_1 \cdots a_{p+1}} = (p+1) \nabla_{[a_1} \omega_{a_2 \cdots a_{p+1}]}$
- Bianchi Identities
  - $R^{a}{}_{[bcd]} = 0$
  - $\nabla_{[e} R^{a}{}_{|b|cd]} = 0$

Levi-Civita Connection

- Definition
  - $\nabla_c g_{ab} = 0$
  - $T^a{}_{bc} = 0$
  - # of eqs = 4 x 10 + 4 x 6 = 64
- Properties
  - $R_{abcd} = R_{cdab}$
- General relativity is described by the Levi-Civita connection.

Connection adapted to a Basis

- Basis and its Dual
  - $(e^{\alpha})_a (e_{\beta})^a = \delta^{\alpha}{}_{\beta}$ $\qquad$ $(e_{\mu})^a (e^{\mu})_b = \delta^a{}_b$
- Definition
  - $\bar{\nabla}_b (e_{\alpha})^a = 0$ implies $\bar{\nabla}_b (e^{\alpha})_a = 0$
- Components of Covariant Derivatives
  - $\bar{\nabla}_c S^a{}_b = (e_{\mu})^a (e^{\nu})_b (e^{\lambda})_c e_{\lambda}(S^{\mu}{}_{\nu})$
- Torsion
  - $\bar{T}^a{}_{bc} = -[e_{\mu}, e_{\nu}]^a (e^{\mu})_b (e^{\nu})_c$
- Riemann Curvature
  - $\bar{R}^a{}_{bcd} = 0$
- Commutation of Connection
  - $(\bar{\nabla}_c \bar{\nabla}_d - \bar{\nabla}_d \bar{\nabla}_c) S^a{}_b = -T^e{}_{cd} \bar{\nabla}_e S^a{}_b$

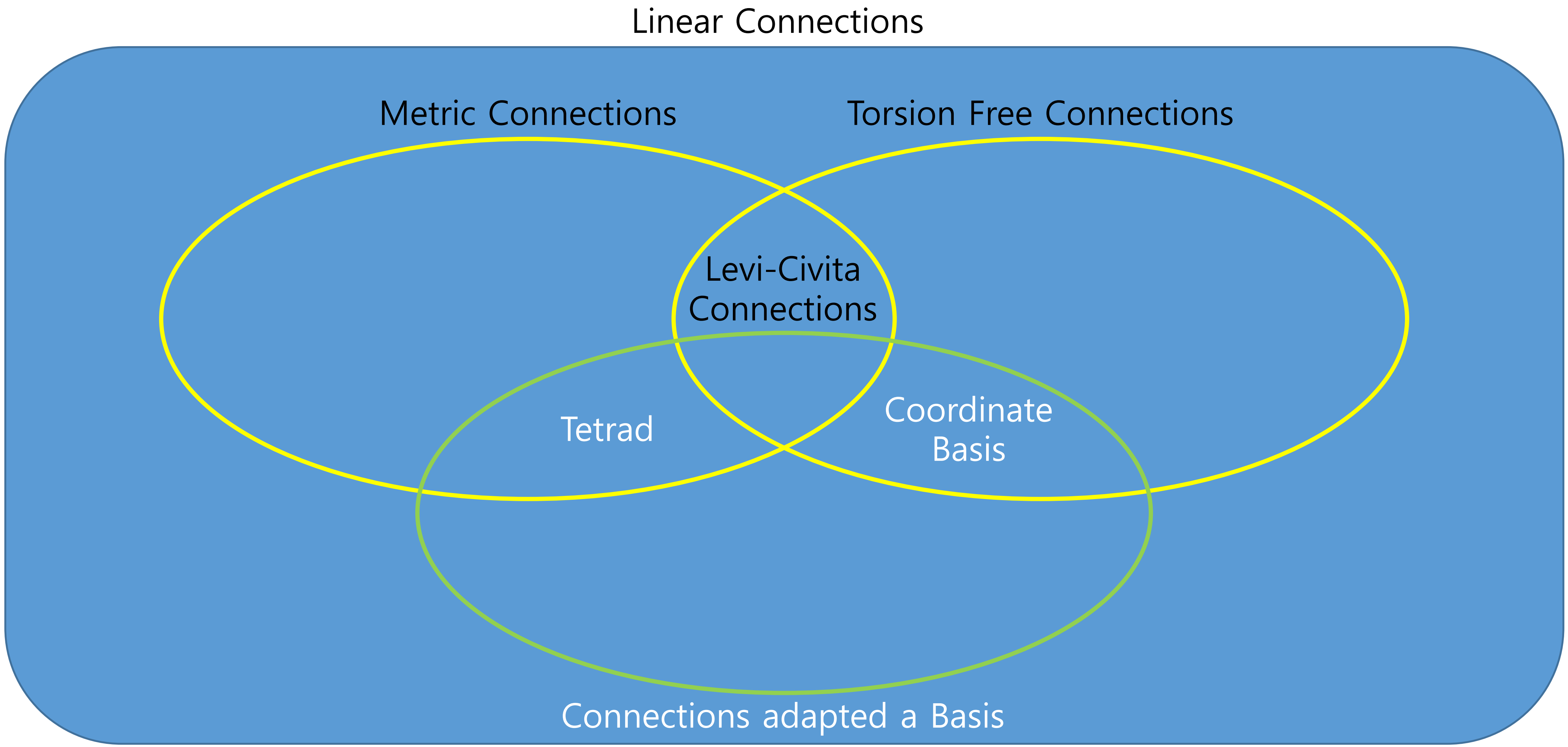
Connection adapted to a Tetrad

- Tetrad (Orthonormal basis)
  - $g_{ab}(e_\alpha)^a(e_\beta)^b = \eta_{\alpha\beta}$
- Metric connection
  - $\bar{\bar{\nabla}}_c g_{ab} = (e^\mu)_a(e^\nu)_b(e^\lambda)_c e_\lambda(\eta_{\mu\nu}) = 0$
  - $\bar{\bar{\nabla}}_e \epsilon_{abcd} = 0$
- We should keep the torsion $\bar{T}^a{}_{bc}$

Connection adapted to a Coordinate Basis

- Torsion Free Connection
  - $\bar{T}^{a}{}_{bc} = -[\partial/\partial x^{\mu}, \partial/\partial x^{\nu}]^{a}(e^{\mu})_{b}(e^{\nu})_{c} = 0$
- Conventionally
  - $\bar{\nabla} \rightarrow \partial$

Summary


Linear Connections
Metric Connections
Torsion Free Connections
Levi-Civita Connections
Tetrad
Coordinate Basis
Connections adapted a Basis

Relation between Two Connections

Connection Relation Tensor

- Definition
  - $(\tilde{\nabla}_c - \nabla_c)S^a{}_b = S^d{}_b C^a{}_{dc} - S^a{}_d C^d{}_{bc}$
- Degree of Freedom for a Connection
  - Given ∇, C completely determines $\tilde{\nabla}$
  - D.O.F of $\tilde{\nabla}$ = D.O.F of C = 4 x 4 x 4 = 64

Relation of Torsions and Riemann Curvatures

- Torsions
  - $\left(\tilde{T}-T\right)^{a}{}_{bc} = -2C^{a}{}_{[bc]}$
- Riemann Curvatures
  - $\left(\tilde{R}-R\right)^{a}{}_{bcd} = 2\nabla_{[c}C^{a}{}_{|b|d]} + 2C^{a}{}_{e[c}C^{e}{}_{|b|d]} + C^{a}{}_{be}T^{e}{}_{cd}$
- Bianchi Identities
  - $\left(\tilde{R}-R\right)^{a}{}_{[bcd]} = -2\nabla_{[b}C^{a}{}_{cd]} - 2C^{a}{}_{e[b}C^{e}{}_{cd]} + T^{e}{}_{[bc}C^{a}{}_{d]e}$
  - $\left(\tilde{\nabla}\tilde{R}-\nabla R\right)^{a}{}_{b[cde]}$ = not yet…

Determination of C

- $C_{abc} = -\frac{1}{2}\left\{\left(\tilde{T} - T\right)_{abc} + \left(\tilde{T} - T\right)_{bca} - \left(\tilde{T} - T\right)_{cab} + \left(\tilde{\nabla} - \nabla\right)_c g_{ab} + \left(\tilde{\nabla} - \nabla\right)_b g_{ca} - \left(\tilde{\nabla} - \nabla\right)_a g_{bc}\right\}$
- Given ∇, $\tilde{\nabla} g$ and $\tilde{T}$ completely determines C
- D.O.F of C = 64 = D.O.F of $\tilde{\nabla} g$ + D.O.F of $\tilde{T}$ = 40 + 24
- Levi-Civita connection is unique because we impose
  - $\nabla_c g_{ab} = 0$ # of Eqs = 40
  - $T^a{}_{bc} = 0$ # of Eqs = 24

Levi-Civita Connection written by Another

- ∇ : Levi-Civita connection
- $\tilde{\nabla}$: Another connection
- $\left(\nabla - \tilde{\nabla}\right)_c S^a{}_b = S^d{}_b C^a{}_{dc} - S^a{}_d C^d{}_{bc}$
- $\tilde{T}^a{}_{bc} = 2C^a{}_{[bc]}$
- $C_{abc} = \frac{1}{2}\left(\tilde{T}_{abc} + \tilde{T}_{bca} - \tilde{T}_{cab} + \tilde{\nabla}_c g_{ab} + \tilde{\nabla}_b g_{ca} - \tilde{\nabla}_a g_{bc}\right)$
- $\left(R - \tilde{R}\right)^a{}_{bcd} = 2\tilde{\nabla}_{[c} C^a{}_{|b|d]} + 2C^a{}_{e[c} C^e{}_{|b|d]} + 2C^a{}_{be} C^e{}_{[cd]}$

Examples

We will apply it to developing 1+3 formalism.

- Written by Metric Connection
  - $C_{abc} = \frac{1}{2}\left(\tilde{T}_{abc} + \tilde{T}_{bca} - \tilde{T}_{cab}\right)$
  - $C_{(ab)c} = 0$
- Written by Torsion Free Connection
  - $C^{a}{}_{[bc]} = 0$
  - $C_{abc} = \frac{1}{2}\left(\tilde{\nabla}_c g_{ab} + \tilde{\nabla}_b g_{ca} - \tilde{\nabla}_a g_{bc}\right)$
  - $\left(R - \tilde{R}\right)^{a}{}_{bcd} = 2\tilde{\nabla}_{[c} C^{a}{}_{|b|d]} + 2C^{a}{}_{e[c} C^{e}{}_{|b|d]}$

Levi-Civita Connection written by Connection adapted to a Basis

- Conventionally,
  - $C \to \Gamma$
- ∇ : Levi-Civita connection
- $\bar{\nabla}$: Connection adapted to a basis
- $(\nabla - \bar{\nabla})_c S^a{}_b = S^d{}_b \Gamma^a{}_{dc} - S^a{}_d \Gamma^d{}_{bc}$
- $\nabla_b (e_\alpha)^a = (e_\alpha)^c \Gamma^a{}_{cb}$ $\qquad$ $\nabla_b (e^\alpha)_a = -(e^\alpha)_c \Gamma^c{}_{ab}$
- $\bar{T}^a{}_{bc} = 2\Gamma^a{}_{[bc]} = -[e_\mu, e_\nu]^a (e^\mu)_b (e^\nu)_c$
- $\Gamma_{abc} = \frac{1}{2}(\bar{T}_{abc} + \bar{T}_{bca} - \bar{T}_{cab} + \bar{\nabla}_c g_{ab} + \bar{\nabla}_b g_{ca} - \bar{\nabla}_a g_{bc})$
- $R^a{}_{bcd} = 2\bar{\nabla}_{[c} \Gamma^a{}_{|b|d]} + 2\Gamma^a{}_{e[c} \Gamma^e{}_{|b|d]} + 2\Gamma^a{}_{be} \Gamma^e{}_{[cd]}$

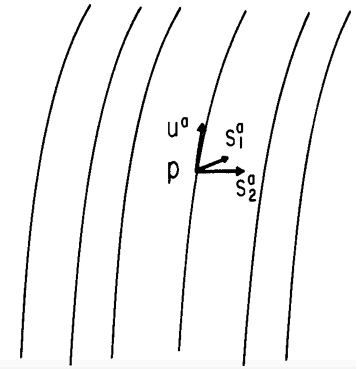
Examples

- Adapted to a tetrad
  - $\Gamma_{abc} = \frac{1}{2}(\bar{T}_{abc} + \bar{T}_{bca} - \bar{T}_{cab})$
    - is the Ricci rotation
  - $\Gamma_{(ab)c} = 0$
- Adapted to a coordinate basis
  - $\Gamma_{a[bc]} = 0$
  - $\Gamma_{abc} = \frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc})$
  - $R^a{}_{bcd} = 2\partial_{[c}\Gamma^a{}_{|b|d]} + 2\Gamma^a{}_{e[c}\Gamma^e{}_{|b|d]}$

Part 2: 1+3 Formalism Development

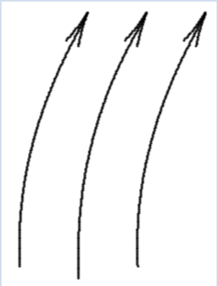
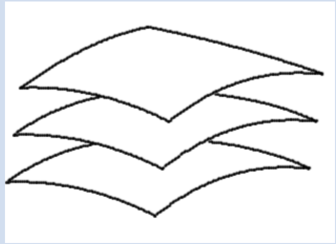
Introduction

Motivation



- In theoretical astrophysics, matter is usually described by perfect fluid which ignores heat transfer and viscosity.
- The perfect fluid provides a congruence of timelike integral curves on spacetime.
- We assume only existence of the timelike congruence for general applicability. Spacetime may not be globally hyperbolic.
- We introduce a time-space splitting formalism with respect to the congruence, i.e. 1+3 formalism.
- 1+3 formalism is applicable to Lagrangian perturbation theory.
- Because we don't have any assumption for the background spacetime, it is applicable to relativistic star, black hole and universe.

Comparison between 1+3 and 3+1

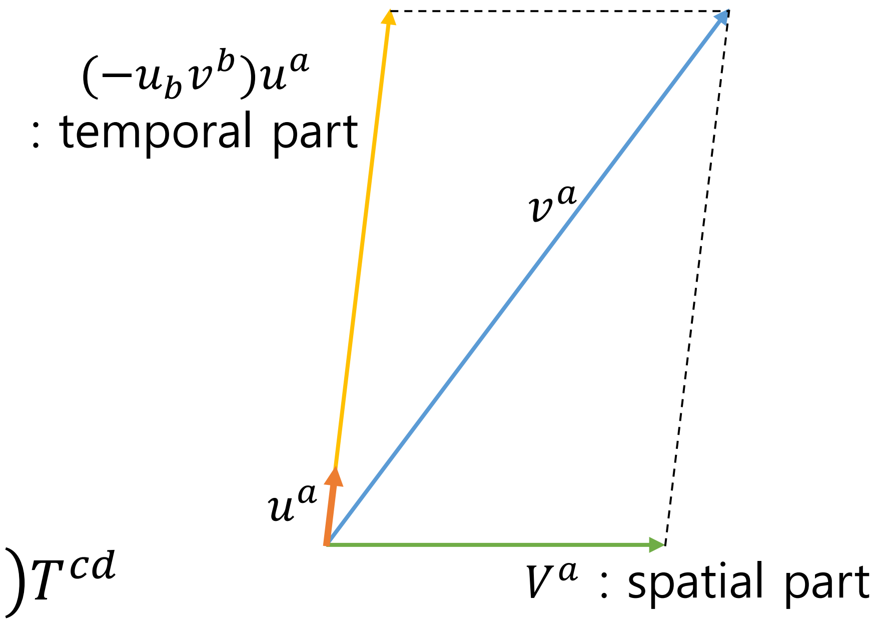
	1+3 Formalism	3+1 Formalism
Given Structure	 u^a : unit tangent vector	 n^a : unit normal vector
Covariant Derivatives	$\nabla_b u_a = -A_a u_b + \frac{1}{3}\Theta\gamma_{ab} + \Sigma_{ab} + \Omega_{ab}$	$\nabla_b n_a = -A_a n_b + K_{ab}$

When $\omega_{ab} = 0$, 1+3 formalism reduces to 3+1 formalism.

Time-Space Splitting

Parallel and Orthogonal Projection

- u^a : unit tangent to the congruence ($g_{ab}u^a u^b = -1$)
- $V^a = v^a - (-u_b v^b)u^a$
  $= (\delta^a{}_b + u^a u_b)v^b$
  $= \gamma^a{}_b v^b$
- Time-space splitting of tensors
  - $v^a = (-u_b v^b)u^a + \gamma^a{}_b v^b$
  - $T^{ab} = (u_c u_d T^{cd})u^a u^b + u^a(-u_c \gamma^b{}_d T^{cd})$
    $+(-\gamma^a{}_c u_d T^{cd})u^b + \gamma^a{}_c \gamma^b{}_d T^{cd}$
- Notation for convenience
  - $\perp(T^{ab}) \equiv \gamma^a{}_c \gamma^b{}_d T^{cd} \quad \parallel(T^{ab}) \equiv (-u^a u_c)(-u^b u_d)T^{cd}$



Splitting of Einstein Equation

- $T_{ab} = \rho u_a u_b + u_a P_b + P_a u_b + p\gamma_{ab} + S_{ab}$
  - Spatial: $P_a u^a = 0$ $S_{ab} u^a = 0$
  - Traceless: $S_{ab} g^{ab} = 0$
- $T = -\rho + 3p$
- Let $8\pi G = 1$
- $R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T$

  $= \bar{\rho} u_a u_b + u_a P_b + P_a u_b + \bar{p}\gamma_{ab} + S_{ab}$
- Where
  - $\bar{\rho} = \frac{1}{2}(\rho + 3p)$ $\bar{p} = \frac{1}{2}(\rho - p)$

Splitting of Weyl Curvature

- $C^{ab}{}_{cd}$: Weyl curvature of Levi-Civita connection associated with spacetime metric g
- $C^{ab}{}_{cd} = 4u^{[a}u_{[c}E^{b]}{}_{d]} + 2u^{[a}H^{b]e}\epsilon_{ecd} + 2u_{[c}H_{d]e}\epsilon^{eab} + 4\gamma^{[a}{}_{[c}E^{b]}{}_{d]}$
- Where
  - $\epsilon_{abc} = u^d\epsilon_{dabc}$
  - $E^a{}_c = C^{ab}{}_{cd}u_b u^d$ $\quad E_{[ab]} = 0 \quad E_{ab}u^a = 0 \quad E_{ab}g^{ab} = 0$
  - $H_{ec} = \frac{1}{2}\epsilon_{eab}C^{ab}{}_{cd}u^d$ $\quad H_{[ab]} = 0 \quad H_{ab}u^a = 0 \quad H_{ab}g^{ab} = 0$

Splitting of Riemann Curvature

- $R^{ab}{}_{cd}$: Riemann curvature of Levi-Civita connection associated with spacetime metric g
- $R^{\ ab}{}_{cd} = C^{ab}{}_{cd} + 2\delta^{[a}{}_{[c}R^{b]}{}_{d]} - \frac{1}{3}R\delta^{[a}{}_{[c}\delta^{b]}{}_{d]}$

$$= 4u^{[a}u_{]c}\left(E^{b]}{}_{d]} + \frac{1}{3}\bar{\rho}\gamma^{b]}{}_{d]} - \frac{1}{2}S^{b]}{}_{d]}\right)$$
$$+2u^{[a}\left(H^{b]e}\epsilon_{ecd} - \gamma^{b]}{}_{[c}P_{d]}\right) + 2u_{[c}\left(H_{d]e}\epsilon^{eab} - \gamma^{[a}{}_{d]}P^{b]}\right)$$
$$+4\gamma^{[a}{}_{[c}\left(E^{b]}{}_{d]} + \frac{1}{12}(\bar{\rho} + 3\bar{p})\gamma^{b]}{}_{d]} + \frac{1}{2}S^{b]}{}_{d]}\right)$$